Worked Out Examples - Concept Based Questions

Example 1: Let **I** be set of integers, **N** = the set of non-negative integers, N_p = the set of non-positive integers. Then the sets A and B satisfying $A \cap B = \phi$ are

(a) $A = I \sim N_p, B = N \sim N_p$
(b) $A = I \sim N, B = I \sim N_p$
(c) $A = N \Delta N_p, B = I \sim N_p$
(d) $A = N \Delta N_p, B = (I \sim N) \cup \{0\}$

Ans. (b)

Solution: $I \sim \mathbf{N} = \{\ldots -3, -2, -1\}$, $I \sim N_p = \{1, 2, 3\ldots\}$, $\mathbf{N} \sim N_p = \{1, 2, \ldots\}$, $N \Delta N_p = (\mathbf{N} \sim N_p) \cup (N_p \sim \mathbf{N})$

$$= \{0, 1, 2, \ldots\}$$

$$(I \sim \mathbf{N}) \cup \{0\} = \{\ldots, -3, -2, -1, 0\}$$

The disjoint sets are $\mathbf{I} \sim \mathbf{N}$ and $\mathbf{I} \sim N_p$.

Example 2: Which of the following equality is not true.

(a) $A \cap (B \sim C) = A \cap B \sim (A \cap C)$
(b) $A \sim (A \cap B) = A \sim B$
(c) $A \sim (B \sim C) = (A \sim B) \cup (A \cap C)$
(d) $A \sim (B \Delta C) = (A \sim B) \Delta (A \sim C)$

Ans. (d)

Solution: For equality (a),

$$\begin{aligned} A \cap (B \sim C) &= A \cap (B \cap C') \\ &= \phi \cup (A \cap B \cap C') \\ &= (A \cap B \cap A') \cup (A \cap B \cap C') \\ &= A \cap B \cap (A' \cup C') \\ &= A \cap B \sim (A \cap C) \end{aligned}$$

For equality (b),

$$\begin{aligned} A \sim (A \cap B) &= A \cap (A' \cup B') \\ &= (A \cap A') \cup (A \cap B') \\ &= \phi \cup (A \cap B') = A \cap B' = A \sim B \end{aligned}$$

For equality (c)

$$\begin{aligned} A \sim (B \sim C) &= A \sim (B \cap C') = A \cap (B' \cup C) \\ &= (A \cap B') \cup (A \cap C) \\ &= (A \sim B) \cup (A \cap C) \end{aligned}$$

For (d) Let $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5\}$, $C = \{1, 2, 3\}$

So, $B \Delta C = \{4, 5\} \cup \{1, 2\} = \{1, 2, 4, 5\}$

Thus $A \sim (B \Delta C) = \{3\}$

$$A \sim B = \{1, 2\}; A \sim C = \{4, 5\}$$

Therefore $(A \sim B) \Delta (A \sim C) = (\{1, 2\} \sim \{4, 5\}) \cup (\{4, 5\} \sim \{1, 2\})$

$$= \{1, 2\} \cup \{4, 5\} = \{1, 2, 4, 5\}.$$

Hence $A \sim (B \Delta C) \neq (A \sim B) \Delta (A \sim C)$

Example 3: A boating club consists of 82 members, each member is either a sailboat owner or a powerboat owner. If 53 members owned sailboats and 38 members owned powerboats, the number of members owned both sailboat and powerboat is

(a) 6 (b) 7
(c) 9 (d) 4

Ans. (c)

Solution: Let S = the set of all members owning sailboats and P = the set of all members owning powerboats

$$n(S \cup P) = n(S) + n(P) - n(S \cap P)$$

$$82 = 53 + 38 - n(S \cap P)$$

$$\Rightarrow \quad n(S \cap P) = 91 - 82 = 9.$$

Example 4: If, $B \underset{\neq}{\subset} A'$, then which of the following sets is equal to A'.

(a) $(A \cap B) \cup B$ (b) $(A \cap B) \cup A'$
(c) $(A \cup B) \cap A'$ (d) $(A \cup B) \cap B$

Ans. (b)

Solution: For (a), $(A \cap B) \cup B = (A \cup B) \cap (B \cup B)$

$$= (A \cup B) \cap B = B.$$

For (b), $(A \cap B) \cup A' = (A \cup A') \cap (B \cup A')$

$$= X \cap A' = A'$$

For (c), $(A \cup B) \cap A' = (A \cap A') \cup (B \cap A')$

$$\begin{aligned} &= \phi \cup (B \cap A') \\ &= B \cap A' = B. \end{aligned}$$

For (d) $(A \cup B) \cap B = B$.

Example 5: If $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] = \frac{5x}{6}$, then x is any term of the following

(a) 3, 6, 9, 12,… (b) 9, 18, 27, 36, …
(c) 6, 12, 18, 24,… (d) $\frac{6}{5}, \frac{12}{5}, \frac{18}{5}, \ldots$

Ans. (c)

◎ **Solution:** Since $\left[\frac{x}{2}\right], \left[\frac{x}{3}\right] \in \boldsymbol{I}$, so $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] \in \boldsymbol{I}$

Thus $\frac{5x}{6} \in \boldsymbol{I} \Rightarrow x = \frac{6}{5}n, n \in \boldsymbol{I}$.

Substituting this value in $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] = \frac{5x}{6}$, we have

$$\left[\frac{3}{5}n\right] + \left[\frac{2}{5}n\right] = n$$

$$\Rightarrow \quad \frac{3}{5}n - \left\{\frac{3}{5}n\right\} + \frac{2}{5}n - \left\{\frac{2}{5}n\right\} = n$$

$$\Rightarrow \quad \left\{\frac{3}{5}n\right\} + \left\{\frac{2}{5}n\right\} = 0$$

$$\Rightarrow \quad \left\{\frac{3}{5}n\right\} = 0 = \left\{\frac{2}{5}n\right\} \qquad \text{(Since } 0 \le \{x\} < 1\text{)}$$

Thus $3n = 5m_1$, $2n = 5m_2$, Therefore $xn = \frac{2n \cdot 3n}{5}$

$$= \frac{5m_1 \times 5m_2}{5} = 5m_1 m_2 \Rightarrow \frac{x \cdot 5m_1}{3} = 5m_1 m_2$$

$$\Rightarrow \quad x = 3m_2$$

Similarly $\frac{x \cdot 5m_2}{2} = 5m_1 m_2 \Rightarrow x = 2m_1$

Hence x is multiple of 2 and 3 so of 6 and $x \in \mathbf{I}$

◉ **Example 6:** The relation R defined by '>' on the set $\mathbf{N}$ is

(a) reflexive (b) symmetric
(c) transitive (d) equivalence relation

Ans. (c)

◎ **Solution:** $2 \not> 2$ so > is not reflexive, $3 > 2$ but $2 \not< 3$ so $(3, 2) \in R$ but $(2, 3) \notin R$. Thus R is not symmetric. If $(a, b) \in R$ and $(b, c) \in R$ then $a > b, b > c \Rightarrow a > c$ so $(a, c) \in R$. R is not an equivalence relation.

◉ **Example 7:** The relation $a\,R\,b$ defined by a is factor of b on $\boldsymbol{N}$ is not

(a) reflexive (b) transitive
(c) anti symmetric (d) symmetric

Ans. (d)

◎ **Solution:** For $a \in \boldsymbol{N}$, a is a factor of a so R is reflexive. If a is factor b and b is factor of c then a is factor of c so R is transitive. If a is factor of b and b is factor of a then $a = b$ so R is anti symmetric, 2 is factor of 4 but 4 is not a factor of 2.

◉ **Example 8:** The domain of the function $f(x) = \log_2 \sin x$ is

(a) $\mathbf{R}$ (b) $\mathbf{R} \sim \{n\pi : n \in \mathbf{I}\}$
(c) $R \sim \{n\pi : n \in \mathbf{N}\}$ (d) $\bigcup_{n \in \mathbf{I}} (2n\pi, (2n+1)\pi)$

Ans. (d)

◎ **Solution:** $f(x) = \log_2 \sin x$ is defined for all x for which $\sin x > 0$. But $\sin x > 0$ if $x \in (0, \pi) \cup (2\pi, 3\pi) \cup \ldots = \bigcup_{n \in \mathbf{I}} (2n\pi, (2n+1)\pi)$.

◉ **Example 9:** The domain of $y = \cos^{-1} \frac{1-2x}{4}$ is

(a) $\left[-\frac{3}{2}, \frac{5}{2}\right]$ (b) $[-1, 1]$
(c) $[0, 2]$ (d) $\left[-\frac{1}{2}, \frac{3}{2}\right]$

Ans. (a)

◎ **Solution:** The given function is defined if

$$-1 \le \frac{1-2x}{4} \le 1 \quad \text{i.e. if } -4 \le 1 - 2x \le 4,$$

$$\Rightarrow -5 \le -2x \le 3 \Rightarrow \frac{-3}{2} \le x \ge \frac{5}{2}$$

◉ **Example 10:** Which of the following functions is bounded

(a) $y = 1 - \log_{10} x$ (b) $y = e^{-2x}$
(c) $y = \sin^{-1}(2x + 1)$ (d) $y = \tan(4x + 1)$

Ans. (c)

◎ **Solution:** The range $\log_{10} x$ is $(-\infty, \infty)$ so $y = 1 - \log_{10} x$ is unbounded. $y = e^{-2x}$ is unbounded from below as $x \to -\infty$, $y \to \infty$. The range of $\sin^{-1} x$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ so $y = \sin^{-1}(2x + 1)$ is a bounded function. The range of $\tan x$ is $\mathbf{R}$ so $y = \tan(4x + 1)$ is unbounded

◉ **Example 11:** A function out of the following whose period is not π is

(a) $\sin^2 x$ (b) $\cos^2 x$
(c) $\tan(2x + 3)$ (d) $y = |\sin x|$

Ans. (c)

◎ **Solution:** $y = \sin^2 x = \frac{1}{2}[1 - \cos 2x]$. Since period of $\cos x$ is 2π so period of $\cos 2x$ is π.

$y = 1 + \cos^2 x = 1 + \frac{1}{2}(1 + \cos 2x)$. The period of this is again π. The period of $\tan x$ is π so period of $\tan(2\pi + 3)$ is $\pi/2$. If $f(x) = |\sin x|$ then $f(x + \pi) = |\sin(x + \pi)| = |-\sin x| = |\sin x| = f(x)$. Thus the period of f is π.

◉ **Example 12:** Which of the following functions is an odd function

(a) $y = x^4 - 2x^2$ (b) $y = x - x^2$
(c) $y = \cos x$ (d) $y = x - \frac{x^3}{6} + \frac{x^5}{40}$

Ans. (d)

◎ **Solution:** If $f(x) = x^4 - 2x^2$ then $f(-x) = (-x)^4 - 2(-x)^2 = x^4 - 2x^2 = f(x)$. Hence f is an even function. If $u(x) = x - x^2$ then $u(-x) = x - x^2$ so u is neither even nor an odd function. $p(x) = \cos x$, $p(-x) = \cos(-x) = \cos x = p(x)$ so p is even function. If $s(x) = x - \frac{x^3}{6} + \frac{x^5}{40}$ then $s(-x) = -x - \frac{x^3}{6} - \frac{x^5}{40} = -\left(x - \frac{x^3}{6} + \frac{x^5}{40}\right) = -s(x)$, so s is an odd function.

Worked Out Examples - Level 1

Example 13: Let $X = \{x : x = n^3 + 2n + 1, n \in \mathbf{N}\}$ and $Y = \{x : x = 3n^2 + 7, n \in \mathbf{N}\}$ then

(a) $X \cap Y$ is a subset of $\{x : x = 3n + 5, n \in \mathbf{N}\}$
(b) $X \cap Y \subseteq \{x : x = n^2 + n + 1, n \in \mathbf{N}\}$
(c) $34 \in X \cap Y$
(d) none of these

Ans. (c)

Solution: If $n^3 + 2n + 1 = 3n^2 + 7$

$\Rightarrow \quad n^3 - 3n^2 + 2n - 6 = 0$

$\Rightarrow \quad (n - 3)(n^2 + 2) = 0$

$\Rightarrow \quad n = 3$ as $n \in \mathbf{N}$

So, $\quad x = 3 \times 3^2 + 7 = 34 \in X \cap Y$.

In (a) and (b) $x \neq 34$, for any $n \in \mathbf{N}$.

Example 14: A, B, C are the sets of letters needed to spell the words STUDENT, PROGRESS and CONGRUENT, respectively. Then $n(A \cup (B \cap C))$ is equal to

(a) 8 (b) 9
(c) 10 (d) 11

Ans. (b)

Solution: $A = \{D, E, N, S, T, U]$
$B = \{E, G, O, P, R, S\}$
$C = \{C, E, G, N, O, R, T, U\}$

So $\quad A \cup (B \cap C)$
$= \{D, E, N, S, T, U\} \cup \{E, G, O, R\}$
$= \{D, E, G, O, N, R, S, T, U\}$

and $n[(A \cup (B \cap C)] = 9$.

Example 15: Let $A = \{x : x$ is a prime factor of $240\}$
$B = \{x : x$ is the sum of any two prime factors of $240\}$.

Then

(a) $5 \notin A \cap B$ (b) $7 \in A \cap B$
(c) $8 \in A \cap B$ (d) $8 \in A \cup B$

Ans. (d)

Solution: $240 = 2 \times 3 \times 5 \times 8$

So $\quad A = \{2, 3, 5\}, \quad B = \{5, 7, 8\}$.

Clearly $8 \in A \cup B$.

Example 16: A, B, C are three sets such that $n(A) = 25$, $n(B) = 20$, $n(c) = 27$, $n(A \cap B) = 5$, $n(B \cap C) = 7$ and $A \cap C = \phi$ then $n(A \cup B \cup C)$ is equal to

(a) 60 (b) 65
(c) 67 (d) 72.

Ans. (a)

Solution: $A \cap C = \phi \quad \Rightarrow \quad A \cap B \cap C = \phi$.

$\therefore\ n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$= 25 + 20 + 27 - 5 - 7 - 0 + 0 = 60$.

Example 17: Let $X = \{(x,y,z) \mid x,y,z \in \mathbf{N}.\ x + y + z = 10, x < y < z\}$ and $Y = \{(x,y,z) \mid x,y,z \in \mathbf{N}, y = |x - z|\}$ then $X \cap Y$ is equal to

(a) $\{(2,3,5)\}$ (b) $\{1,4,5\}$
(c) $\{5,1,4\}$ (d) $\{(2,3,5), (1,4,5)\}$

Ans. (d)

Solution: $X = \{(1,2,7), (1,3,6), (1,4,5), (2,3,5)\}$.
Elements of X which belong to Y are (1,4,5) and (2,3,5) both so they belong to $X \cap Y$.

Example 18: If A,B,C are three non-empty sets such that $A \cap B = \phi$, $B \cap C = \phi$, then

(a) $A = C$ (b) $A \subset C$
(c) $C \subset A$ (d) none of these

Ans. (d)

Solution: Let $A = \{1,2,3,4,5\}$, $B = \{6,7,8,9\}$ and $C = \{11,12,13\}$ which satisfy the given conditions but none of (a), (b) or (c).

Example 19: Two finite sets have m and n elements respectively. The total number of subsets of first set is 56 more than the total number of subsets of the second set. The values of m and n respectively are

(a) 7, 6 (b) 6, 3
(c) 5, 1 (d) 8, 7

Ans. (b)

Solution: According to the given condition, we have $2^m = 2^n + 56$
$\Rightarrow 2^{m-3} - 2^{n-3} = 7 \Rightarrow 2^{n-3}(2^{m-n} - 1) = 7$. Since 7 is a prime number so we must have $n - 3 = 0$ (clearly $m \neq n$). Thus $n = 3$. Therefore, $2^m = 2^3 + 56 = 64 = 2^6 \Rightarrow m = 6$.

Example 20: Among employee of a company taking vacations last years, 90% took vacations in the summer, 65% in the winter, 10% in the spring, 7% in the autumn, 55% in winter and summer, 8% in the spring and summer, 6% in the autumn and summer, 4% in the winter and spring, 4% in winter and autumn, 3% in the spring and autumn, 3% in the summer, winter and spring, 3% in the summer, winter and autumn, 2% in the summer, autumn and spring, and 2% in the winter, spring and autumn. Percentage of employee that took vacations during every season:

(a) 4 (b) 3
(c) 2 (d) 8

Ans. (c)

◎ **Solution:** Suppose that number of employee taking vacations is 100.

Su – set of employee taking leave in Summer
W – set of employee taking leave in Winter
Sp – set of employee taking leave in Spring
A – set of employee taking leave in Autumn

$n(Su) = 90, n(W) = 65, n(Sp) = 10, n(A) = 7$

$n(W \cap Su) = 55, n(Sp \cap Su) = 8, n(A \cap Su) = 6$

$n(W \cap Sp) = 4, n(W \cap Au) = 4, n(Sp \cap A) = 3$

$n(Su \cap A) = 3, n(Su \cap W \cap A) = 3$

$n(Su \cap W \cap Sp) = 3, n(Su \cap A \cap Sp) = 2$

$n(W \cap Sp \cap A) = 2$

$$\begin{aligned} n(Su \cap Sp \cap W \cap A) &= n(Su) + n(Sp) + n(W) + n(A) - n(Su \cap Sp) \\ &- n(Sp \cap W) - n(W \cap A) - n(Su \cap A) - n(Su \cap W) \\ &- n(Sp \cap A) + n(Su \cap Sp \cap W) + n(Su \cap W \cap A) \\ &+ n(W \cap A \cap Su) + n(Su \cap Sp \cap A) \\ &- n(Sp \cup Su \cup A \cup W) \\ &= 90 + 65 + 10 + 7 - 55 - 8 - 6 - 4 - 4 - 3 \\ &+ 3 + 3 + 2 + 2 - 100 = 2 \end{aligned}$$

◉ **Example 21:** If $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$ then the number of elements in $(A \cup B) \times (A \cap B) \times (A \Delta B)$ is

(a) 5 (b) 30
(c) 10 (d) 4

Ans. (b)

◎ **Solution:** $A \cup B = \{1, 2, 3, 4, 5\}$. $n(A \cup B) = 5$

$A \cap B = \{3, 4\}, n(A \cap B) = 2$

$A \Delta B = (A \sim B) \cup (B \sim A) = \{1, 2\} \cup \{5\} = \{1, 2, 5\}$

$n(A \Delta B) = 3$. Hence $n((A \cup B) \times (A \cap B) \times (A \Delta B)\} = 5 \times 2 \times 3 = 30$.

◉ **Example 22:** Let **I** be the set of integers. For $a, b \in \mathbf{I}$, $a\,R\,b$ if and only if $|a - b| < 1$, then

(a) R is not reflexive
(b) R is not symmetric
(c) $R = \{(a,a); a \in \mathbf{I}\}$
(d) R is not an equivalence relation.

Ans. (c)

◎ **Solution:** For any integers a, b, $|a - b| < 1$ if and only if $|a - b| = 0$ so $a = b$. Hence $R = \{(a,a); a \in \mathbf{I}\}$.Thus R is reflexive, symmetric and transitive.

◉ **Example 23:** Let W denote the words in the English Dictionary. Define the relation **R** by $\mathbf{R} = \{(x, y) \in W \times W :$ the words x and y have at least one letter common$\}$, then **R** is

(a) reflexive, not symmetric and transitive
(b) not reflexive, symmetric and transitive
(c) reflexive, symmetric and not transitive
(d) reflexive, symmetric and transitive

Ans. (c)

◎ **Solution:** $(x, x) \in \mathbf{R} \quad \forall\ x \in W$ as all letters in both are common. If $(x, y) \in \mathbf{R}$ then x and y have a letter in common $\Rightarrow (y, x) \in \mathbf{R}$.

Next, let x = fix, y = six and z = son then$(x, y) \in \mathbf{R}$, $(y, z) \in \mathbf{R}$ but $(x, z) \notin \mathbf{R}$

So **R** is reflexive, symmetric but not transitive

◉ **Example 24:** If the relation $R: A \to B$, where $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5\}$ is defined by $R = \{(x, y); x < y, x \in A, y \in B\}$ then RoR^{-1} is

(a) $\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$
(b) $\{(3, 1), (5, 1), (5, 2), (5, 3), (5, 4)\}$
(c) $\{(3, 3), (3, 5), (5, 3), (5, 5)$
(d) none of these

Ans. (c)

◎ **Solution:** $R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$

and $R^{-1} = \{(3, 1), (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)\}$.

Thus $RoR^{-1} = \{(3, 3), (3, 5), (5, 3), (5, 5)\}$.

◉ **Example 25:** Let $A = \{x \in \mathbf{R} : [x + 3] + [x + 4] \le 3\}$ and $B = \left\{x \in \mathbf{R} : 3^x \left(\sum_{r=1}^{\infty} \frac{3}{10^r}\right)^{x-3} < 3^{-3x}\right\}$ then

(a) $A = B$ (b) $A \subsetneq B$
(c) $B \subsetneq A$ (d) $A \cap B = \phi$

Ans. (a)

◎ **Solution:** Let $x \in A$, $[x + 3] + [x + 4] \le 3$

$$\begin{aligned} &\Rightarrow \quad [x] + 3 + [x] + 4 \le 3 \\ &\Rightarrow \quad 2[x] \le -4 \ \Rightarrow \ [x] \le -2 \\ &\qquad\qquad \Rightarrow \ x \in (-\infty, -1) \\ &\quad A = (-\infty, -1) \end{aligned}$$

If $x \in B$ then $3^x\, 3^{x-3} \left(\sum_{r=1}^{\infty} \frac{1}{10^r}\right)^{x-3} < 3^{-3x}$

$$\Rightarrow \quad 3^{2x-3}\left(\frac{1/10}{1 - 1/10}\right)^{x-3} < 3^{-3x}$$

$$\Rightarrow \quad 3^{2x-3}\left(3^{-2}\right)^{x-3} < 3^{-3x}$$

$$\Rightarrow \quad 3^3 < 3^{-3x} \ \Rightarrow \ 3 < -3x$$

so $x \in (-\infty, -1)$

Hence $B = (-\infty, -1)$. Thus $A = B$.

◉ **Example 26:** The range of the function $f(x) = {}^{7-x}P_{x-3}$ is

(a) $\{1, 2, 3, 4\}$ (b) $\{1, 2, 3, 4, 5, 6\}$
(c) $\{1, 2, 3\}$ (d) $\{1, 2, 3, 4, 5\}$

Ans. (c)

Solution: $7 - x \geq 1$, $x - 3 \geq 0$

and $\quad 7 - x \geq x - 3$

$\Rightarrow \quad x \leq 6, x \geq 3, x \leq 5.$

Thus $\quad 3 \leq x \leq 5$

$\therefore \quad$ Range $= \{4P_0, 3P_1, 2P_2\}$

$= \{1, 3, 2\}$

Example 27: The domain of the function

$f(x) = \dfrac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is

(a) [1, 2] (b) [2, 3)
(c) [1, 3) (d) [1, 2]

Ans. (b)

Solution: $-1 \leq x - 3 \leq 1$ and $9 - x^2 > 0$

$\Rightarrow \quad 2 \leq x \leq 4$ and $-3 < x < 3$.

So domain of f is [2, 3).

Example 28: The solution of $8x \equiv 6 \pmod{14}$ is

(a) [8], [6] (b) [6], [14]
(c) [6], [13] (d) [8], [14], [16]

where $[a] = \{a + 14\,k : k \in \mathbf{I}\}$

Ans. (c)

Solution: We need to solve $14|(8x - 6)$ i.e., $8x - 6 = 14k$, for $k \in I$. The values 6 and 13 satisfy this equation, while 8, 14 and 16 do not.

Example 29: Let $R = \{(x, y): x, y \in A, x + y = 5\}$, where $A = \{1, 2, 3, 4, 5\}$ then

(a) R is not reflexive, symmetric and not transitive
(b) R is an equivalence relation
(c) R is reflexive, symmetric but not transitive
(d) R is not reflexive, not symmetric but transitive

Ans. (a)

Solution: $R = \{(1, 4), (4, 1), (2, 3), (3, 2)\}$, so R is not reflexive as $(1, 1) \notin R$. R is symmetric by definition and R is not transitive as $(1, 4) \in R$, $(4, 1) \in R$ but $(1, 1) \notin R$.

Example 30: Let R be a relation on a set A such that $R = R^{-1}$ then R is

(a) reflexive (b) symmetric
(c) transitive (d) an equivalence relation

Ans. (b)

Solution: If $(a, b) \in R$ then $(b, a) \in R^{-1} = R$ so R is symmetric. The relation in Example 29 satisfy $R = R^{-1}$ but is neither reflexive nor transitive.

Example 31: For $x, y \in \mathbf{R}$, define a relation R by $x\,R\,y$ if and only if $x - y + \sqrt{2}$ is an irrational number. Then R is

(a) reflexive (b) symmetric
(c) transitive (d) none of these

Ans. (a)

Solution: Since $x - x + \sqrt{2} = \sqrt{2}$ which is an irrational number so $x\,R\,x$ for all $x \in \mathbf{R}$. Hence R is reflexive. R is not symmetric as $(\sqrt{2}, 1) \in R$ but $(1, \sqrt{2}) \notin R$. Again R is not transitive since $(\sqrt{2}, 1) \in R$ and $(1, 2\sqrt{2}) \in R$ but $(\sqrt{2}, 2\sqrt{2}) \notin R$.

Example 32: If $n|q$ and $A = \{z \in C : z^n = 1\}$, $B = \{z : z^q = 1\}$ then

(a) $A = B$ (b) $A \cap B = \{1\}$
(c) $B \subseteq A$ (d) $A \subsetneq B$

Ans. (d)

Solution: $q = p\,n$ for some $p \in \mathbf{N}$

$$z^q - 1 = (z^n)^p - 1 = (z^n - 1)\,(z^{(p-1)n} + \cdots + z^n + 1)$$

Every root of $z^n - 1$ is a root of $z^q - 1$ and every root of $z^{(p-1)n} + \cdots + z^n + 1 = 0$ is also a root of $z^q - 1$. Hence $A \subsetneq B$ and $A \cap B = A$.

Example 33: If $A = \{z : (1 + 2i)\,\bar{z} + (1 - 2i)\,z + 2 = 0\}$ and $B = \{z : (3 + 2i)\,\bar{z} + (3 - 2i)\,z + 3 = 0\}$, then

(a) $A \cap B$ is a singleton set
(b) $A \subseteq B$
(c) $B \subseteq A$
(d) $A \cap B = \phi$.

Ans. (a)

Solution: Equations in sets A and B represent straight line with $\bar{\alpha}_1 = 1 + 2i$ and $\bar{\alpha}_2 = 3 + 2i$. Since $\dfrac{\bar{\alpha}_1}{\alpha_1} \neq \dfrac{\bar{\alpha}_2}{\alpha_2}$ so the lines are intersecting, hence $A \cap B$ is a singleton set.

Example 34: Let $x, y \in \mathbf{I}$ and suppose that a relation R on $\mathbf{I}$ is defined by $x\,R\,y$ if and only if $x \leq y$ then

(a) R is reflexive but not symmetric
(b) R is an equivalence relation
(c) R is neither reflexive nor symmetric
(d) R is symmetric but not transitive

Ans. (a)

Solution: Since $x \leq x$ for all $x \in \mathbf{I}$ so R is reflexive but is not symmetric as $(1, 2) \in R$ and $(2, 1) \notin R$. Also R is transitive as $x \leq y$, $y \leq z \Rightarrow x \leq z$.

Example 35: If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 + 1$, then value of $f^{-1}(17)$ and $f^{-1}(-3)$ are, respectively,

(a) ϕ, {4, –4} (b) ϕ, {3, –3}
(c) {3, –3}, ϕ (d) {4, –4}, ϕ

Ans. (d)

Solution: For any $A \subseteq \mathbf{R}$, we have $f^{-1}(A) = \{x \in \mathbf{R}: f(x) \in A\}$. Thus,

$f^{-1}(17) = \{x : f(x) \in \{17\}\} = \{x : f(x) = 17\}$

$= \{x : x^2 + 1 = 17\} = \{4, -4\}$,

and similarly, $f^{-1}(-3) = \{x \in \mathbf{R} : x^2 + 1 = -3\} = \phi$.

Example 36: The functions f and g are given by $f(x) = \{x\}$, the fractional part of x and $g(x) = \frac{1}{2}\sin[x]\pi$, where $[x]$ denotes the integral part of x. Then range of gof is

(a) $[-1, 1]$ (b) $\{0\}$
(c) $\{-1, 1\}$ (d) $[0, 1]$

Ans. (b)

Solution: $(gof)(x) = g(f(x)) = 1/2 \sin[\{x\}]\pi = 0$, for all $x \in \mathbf{R}$. Hence the range of gof is $\{0\}$.

Example 37: The period of the function $f(x) = \cos^2 3x + \tan 4x$ is

(a) $\pi/3$ (b) $\pi/4$
(c) $\pi/6$ (d) π

Ans. (d)

Solution: $f(x) = (1/2)(1 + \cos 6x) + \tan 4x$. The period of $\cos 6x$ is $2\pi/6 = \pi/3$ and the period of $\tan 4x$ is $\pi/4$. Hence the period of f is l.c.m. of $\pi/3$ and $\pi/4 = \pi$.

Example 38: The domain of the function

$$f(x) = \sin^{-1}\left(\log_3 \frac{x}{3}\right) \text{ is}$$

(a) $[-1, 9]$ (b) $[1, 9]$
(c) $[-9, 1]$ (d) $[3, 9]$

Ans. (b)

Solution: The function f is defined only if $-1 \le \log_3 (x/3) \le 1$. This inequality is possible only if $1/3 \le x/3 \le 3$ i.e., $1 \le x \le 9$.

Example 39: The domain of the function

$$f(x) = \frac{\sqrt{-\log_{0.3}(x-1)}}{\sqrt{-x^2+2x+8}} \text{ is}$$

(a) $(1, 4)$ (b) $(-2, 4)$
(c) $(2, 4)$ (d) none of these

Ans. (c)

Solution: Since for, $0 < a < 1$, $\log_a x < 0$ for $x > 1$ so $\log_{0.3}(x - 1) < 0$ for $x > 2$. Also $-x^2 + 2x + 8 > 0$ if and only if $x \in (-2, 4)$. Hence the domain of the given function is $(2, 4)$.

Example 40: The function f: $[-1/2, 1/2] \to [-\pi/2, \pi/2]$ defined by $f(x) = \sin^{-1}(3x - 4x^3)$ is

(a) both one-one and onto
(b) neither one-one nor onto
(c) onto but not one-one
(d) one-one but not onto

Ans. (a)

Solution: Since $\sin^{-1}(3x - 4x^3) = 3\sin^{-1} x \in [-\pi/2, \pi/2]$ i.e., $\sin^{-1} x \in [-\pi/6, \pi/6]$ or $x \in [-1/2, 1/2]$ so f is onto. Also $f'(x) = \frac{3}{\sqrt{1-x^2}} > 0$ for $-1/2 < x < 1/2$. Therefore, f increases on $[-1/2, 1/2]$ and hence f is one-one.

Example 41: Given $f(x) = \frac{1}{\sqrt{|x| - x}}$ and $g(x) = \frac{1}{\sqrt{x - |x|}}$ then

(a) dom $f \ne \phi$ and dom $g = \phi$
(b) dom $f = \phi$ and dom $g \ne \phi$
(c) f and g have the same domain
(d) dom $f = \phi$ and dom $g = \phi$

Ans. (a)

Solution: dom $f = \{x : |x| > x\}$ and dom $g = \{x : x > |x|\} = \phi$. Thus dom $f = \mathbf{R}^-$ (the set of all negative real numbers) and dom $g = \phi$.

Example 42: Which of the following functions is not onto

(a) f: $\mathbf{R} \to \mathbf{R}$, $f(x) = 3x + 4$
(b) f: $\mathbf{R} \to \mathbf{R}^+$, $f(x) = x^2 + 2$
(c) f: $\mathrm{R}^+ \to \mathrm{R}^+$, $f(x) = \sqrt{x}$
(d) none of these

Ans. (b)

Solution: The function $f(x) = 3x + 4$ is onto as for $y \in \mathbf{R}$, $f\left(\frac{y-4}{3}\right) = y$. The function $f : \mathbf{R}^+ \to \mathbf{R}^+$, $f(x) = \sqrt{x}$ is onto as for $y \in \mathbf{R}^+$, $f(y^2) = y$. $f : \mathbf{R} \to \mathbf{R}^+$, $f(x) = x^2 + 2$ is not onto as $1 \in \mathbf{R}^+$ has no pre-image.

Example 43: Which of the following functions is not one-one

(a) f: $\mathbf{R} \to \mathbf{R}$, $f(x) = 2x + 5$
(b) f: $[0, \pi] \to [-1, 1]$, $f(x) = \cos x$
(c) f: $[-\pi/2, \pi/2] \to [1, 7]$ $f(x) = 3\sin x + 4$
(d) $f : \boldsymbol{R} \to [-1, 1]$, $f(x) = \sin x$

Ans. (d)

Solution: The function in (*a*) is one-one as $2x_1 + 5 = 2x_2 + 5$ is possible only if $x_1 = x_2$. The function in (*b*) is one-one as $\cos x_1 = \cos x_2$ if and only if $\sin\frac{x_1 - x_2}{2} = 0$ i.e., $x_1 = x_2$. Similarly the function in (*c*) is also one-one. The function in (*d*) is not one-one as $f(\pi) = f(2\pi) = 0$.

Example 44: Which of the following functions is non-periodic

(a) $f(x) = x - [x]$
(b) $f(x) = \begin{cases} 1 \text{ if } x \text{ is a rational number} \\ 0 \text{ if } x \text{ is an irrational number} \end{cases}$
(c) $f(x) = \sqrt{\frac{8}{1+\cos x} + \frac{8}{1-\cos x}}$
(d) none of these

Ans. (d)

◎ **Solution:** The function in (a) is periodic with period 1 and the function in (b) is also periodic since $f(x + r) = f(x)$ for every rational r. The function in (c) is equal to $\frac{4}{|\sin x|}$ and thus has period π.

◉ **Example 45:** Let $f(x) = \frac{\alpha x}{x+1}$, $x \neq -1$. Then, for what value of α is $f(f(x)) = x$?

(a) $\sqrt{2}$ (b) $-\sqrt{2}$
(c) 1 (d) –1

Ans. (d)

◎ **Solution:** $f(f(x)) = \frac{\alpha f(x)}{f(x)+1} = \frac{\alpha^2 x}{(\alpha+1)x+1}$

Thus $f(f(x)) = x \Leftrightarrow \alpha^2 x = (\alpha + 1)x^2 + x$

$\Leftrightarrow (\alpha^2 - 1)x = (\alpha + 1)x^2$

$\Leftrightarrow (\alpha + 1)((\alpha - 1) - (\alpha + 1)x) = 0$

Since $(\alpha - 1) - (\alpha + 1)x \neq 0$ for all x so $\alpha = -1$.

◉ **Example 46:** Let $R = \{(x, y): x, y \in \mathbf{R}, x^2 + y^2 \leq 25\}$

$R' = \left\{(x, y): x, y \in \mathbf{R}, y \geq \frac{4}{9}x^2\right\}$ then

(a) dom $R \cap R' = [-4, 4]$
(b) range $R \cap R' = [0, 4]$
(c) range $R \cap R' = [0, 5]$
(d) $R \cap R'$ defines a function.

Ans. (c)

◎ **Solution:** The equation $x^2 + y^2 = 25$ represents a circle with centre (0, 0) and radius 5 and the equation $y = \frac{4}{9}x^2$ represents a parabola with vertex (0, 0) and focus (0, 1/9). Hence $R \cap R'$ is the set of points indicated in the Fig.1.27

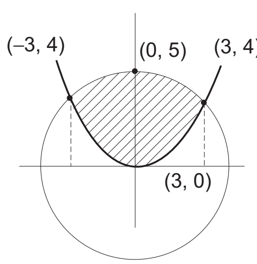

Fig. 1.27

$= \{(x, y): -3 \leq x \leq 3, 0 \leq y \leq 3]\}$. Thus
dom $R \cap R' = [-3, 3]$ and range $R \cap R' = [0, 5] \supset [0, 4]$
Since $(0, 0) \in R \cap R'$ and $(0, 5) \in R \cap R'$
$\therefore$ 0 is related to 0 as well as 5.
Hence $R \cap R'$ doesn't define a function.

◉ **Example 47:** In a factory 70% of the workers like oranges and 64% like apples. If x% like both oranges and apples, then

(a) $x \leq 34$ (b) $x \geq 64$
(c) $34 \leq x \leq 64$ (d) none of these.

Ans. (c)

◎ **Solution:** Let the total number of workers be 100. A, the set of workers who like oranges and B, the set of workers who like apples.

So $n(A) = 70, n(B) = 64, n(A \cap B) = x$.
Also $n(A \cup B) \leq 100$.
$\Rightarrow$ $n(A) + n(B) - n(A \cap B) \leq 100$
$\Rightarrow$ $70 + 64 - x \leq 100 \Rightarrow x \geq 34$
Since $n(A \cap B) \leq n(B) \Rightarrow x \leq 64$
Hence $34 \leq x \leq 64$.

◉ **Example 48:** The Cartesian product of $A \times A$ has 16 elements. $S = \{(a, b) \in A \times A | a < b\}$. (– 1, 2) and (0, 1) are two elements belonging to S. The remaining elements of S are given by.

(a) {(–1, 0), (– 1, 1), (0, 2), (1, 2)}
(b) {(–1, 0), (1, 1), (2, –1), (1, 2)}
(c) {(0, –1), (1, – 1), (0, 2), (1, 2)}
(d) none of these

Ans. (a)

◎ **Solution:** $(-1, 2) \in A \times A$
$\Rightarrow$ $-1 \in A$, $2 \in A$ and $(0, 1) \in A \times A \Rightarrow 0 \in A, 1 \in A$
So, $A = \{-1, 0, 1, 2\}$ as A has four elements
and $S = \{(-1, 0), (-1, 1), (-1, 2), (0, 1), (0, 2), (1, 2)\}$.
Hence the required element of S are given by (a)

◉ **Example 49:** If R and R' are two symmetric relations (not disjoint) on a set A, then the relation $R \cap R'$ is

(a) reflexive (b) symmetric
(c) transitive (d) none of these.

Ans. (b)

◎ **Solution:**
Let $(a, b) \in R \cap R'$ for some $a, b \in A$.
$\Rightarrow$ $(a, b) \in R$ and $(a, b) \in R'$
$\Rightarrow$ $(b, a) \in R$ and $(b, a) \in R'$ as R and R' are symmetric
$\Rightarrow$ $(b, a) \in R \cap R'$ showing that $R \cap R'$ is symmetric.

◉ **Example 50:** α, β, γ denote respectively the sets containing the letters in the names Apoorv, Mannan and Manvi of three children playing together. Which of the following is not correct.

(a) $n(\alpha \cap \gamma) | n(\alpha \cup \beta \cup \gamma)$
(b) $n(\beta \cap \gamma) | n(\alpha \cup \beta \cup \gamma)$
(c) $n(\alpha \cup \beta \cup \gamma) = 8$
(d) $n(\alpha \cup \beta \cup \gamma) = n(\alpha \cup \gamma)$

Ans. (b)

◎ **Solution:** $\alpha = \{a, o, p, r, v\}$, $\beta = \{a, m, n\}$
$\gamma = \{a, i, m, n, v\}$
$\alpha \cap \beta = \{a\}$, $\alpha \cap \gamma = \{a, v\}$, $\beta \cap \gamma = \{a, m, n\}$
$\alpha \cup \beta \cup \gamma = \{a, i, m, n, o, p, r, v\} = \alpha \cup \gamma$.

Example 51: If $f : \mathbf{R} \to \mathbf{R}$, defined by $f(x) = x^3 + 7$, then the value of $f^{-1}(71)$ and $f^{-1}(-1)$ respectively are

(a) $\{4\}, \phi$ (b) $\phi, \{-2\}$
(c) $\{4\}, \{-2\}$ (d) $\{2\}, \{-4\}$

Ans (c)

Solution: $f(x) = x^3 + 7 = 71 \Rightarrow x^3 = 64 \Rightarrow x = 4$

$\Rightarrow \quad f^{-1}(71) = \{4\}$

and $f(x) = x^3 + 7 = -1 \Rightarrow x^3 = -8 \Rightarrow x = -2 \Rightarrow f^{-1}(-1) = \{-2\}$

Example 52: If $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$, $B = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$ and $\mathbf{N}$, the set of natural numbers is the universal set, then $A' \cup [(A \cup B) \cap B']$ is

(a) A (b) A'
(c) B (d) $\mathbf{N}$.

Ans. (d)

Solution: $(A \cup B) \cap B' = A$ as $A \cap B = \phi$

$\Rightarrow \quad A' \cup [(A \cup B) \cap B'] = A' \cup A = \mathbf{N}$.

Example 53: If $X = \{1, 2, 3, 4,\}$, then a one-one onto mapping $f: X \to X$ such that $f(1) = 1$, $f(2) \neq 2$ and $f(4) \neq 4$ is given by

(a) $\{(1, 1), (2, 3), (3, 4), (4, 2)\}$
(b) $\{(1, 1), (2, 4), (3, 1), (4, 2)\}$
(c) $\{(1, 2), (2, 4), (3, 2), (4, 3)\}$
(d) none of these

Ans. (a)

Solution: f in (a) is clearly one-one and onto also satisfies $f(1) = 1$, $f(2) = 3 \neq 2$, $f(4) = 2 \neq 4$.

Example 54: Let $f(x) = (x + 1)^2 - 1$, $x \geq -1$ then the set $\{x : f(x) = f^{-1}(x)\}$ is equal to

(a) $\{0, -1\}$ (b) $\{0, 1\}$
(c) $\{-1, 1\}$ (d) $\{0\}$

Ans. (a)

Solution: Let $y = (x + 1)^2 - 1$, $x \geq -1$

$\Rightarrow \quad (x + 1)^2 = y + 1$

$\Rightarrow \quad x + 1 = \sqrt{y+1}$ as $x \geq -1$

$\Rightarrow \quad x = -1 + \sqrt{y+1}$, $y \geq -1$

Thus $\quad f^{-1}(x) = -1 + \sqrt{x+1}$

So $\quad f(x) = f^{-1}(x)$

$\Rightarrow \quad (x + 1)^2 - 1 = -1 + \sqrt{x+1}$

$\Rightarrow \quad \sqrt{x+1} = 0$ or $(x + 1)^{3/2} = 1$

$\Rightarrow \quad x = -1$ or $x = 0$

Example 55: Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets, then

(a) $P \subset Q$ and $Q \sim P \neq \phi$ (b) $Q \not\subset P$
(c) $P \not\subset Q$ (d) $P = Q$.

Ans. (d)

Solution: $\sin\theta - \cos\theta = \sqrt{2}\cos\theta$

$\Leftrightarrow \quad \sin\theta = (\sqrt{2} + 1)\cos\theta$

$\Leftrightarrow \quad \cos\theta = (\sqrt{2} - 1)\sin\theta$

$\Leftrightarrow \quad \sin\theta + \cos\theta = \sqrt{2}\sin\theta$

$\Rightarrow \quad P = Q$.

Example 56: If A, B, C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then

(a) $B = C$ (b) $A \cap B = \phi$
(c) $A = B$ (d) $A = C$

Ans. (a)

Solution: Let $x \in B \Rightarrow x \in A \cup B$

$\Rightarrow \quad x \in A \cup C \Rightarrow x \in A$ or $x \in C$

If $x \in A$, then $x \in A \cap B = A \cap C \Rightarrow x \in C$

So $\quad x \in B \Rightarrow x \in C$.

Similarly $\quad x \in C \Rightarrow x \in B$, Hence $B = C$.

Example 57: The domain of the function

$$f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}} \text{ is}$$

(a) [1, 2] (b) [2, 3]
(c) [1, 3] (d) [1, 2]

Ans. (b)

Solution: $x^2 < 9 \Rightarrow -3 < x < 3$

and $-1 \leq x - 3 \leq 1 \Rightarrow 2 \leq x < 3$

Example 58: Let $X = \{1, 2, 3, 4, 5\}$. The number of different ordered pairs (Y, Z) that can be formed such that $Y \subseteq X$, $Z \subseteq X$, and $Y \cap Z$ is empty is

(a) 2^5 (b) 5^3
(c) 5^2 (d) 3^5

Ans. (d)

Solution: For each $x \in X$, we have three choices

$x \in Y, x \notin Z$; $\quad x \notin Y, x \in Z$; $\quad x \notin Y, x \notin Z$

So the required number of ordered pairs is 3^5.

Example 59: Let $X = \{1, 2, 3, 4\}$. The number of equivalence relations that can be defined on X is

(a) 10 (b) 15
(c) 16 (d) 8

Ans. (b)

Solution: The number of equivalence relations

$$B_k = \sum_{n=0}^{k-1} \binom{k-1}{n} B_n; \quad B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 5$$

$$B_4 = \binom{3}{0}B_0 + \binom{3}{1}B_1 + \binom{3}{2}B_2 + \binom{3}{3}B_3$$

$$= 1 + 3 + 3 \times 2 + 1 \times 5 = 15$$

Example 60: The function $f(x) = \sin\dfrac{\pi x}{n!} - \cos\dfrac{\pi x}{(n+1)!}$ is

(a) non periodic
(b) periodic with period $2(n!)$
(c) periodic with period $2\,(n+1)!$
(d) periodic with period $n!$

Ans. (c)

Solution: Since the period of $\sin x$ is 2π so the period of $\sin\dfrac{\pi x}{n!}$ is $\dfrac{2\pi n!}{\pi} = 2(n!)$. The period of $\cos\dfrac{\pi x}{(n+1)!}$ is $2(n+1)!$ The period of $f(x) =$ l.c.m. $(2(n!), 2(n+1)!) = 2(n+1)!$

Example 61: If $f(x)^2\ f\left(\dfrac{1-x}{1+x}\right) = x^3,\ \ x \neq -1,\ 1$ and $f(x) \neq 0$, then $\{f(-2)\}$ (the fractional part of $f(-2)$) is equal to

(a) $2/3$ (b) $1/3$
(c) $1/2$ (d) 0

Ans. (a)

Solution: Replacing x by $\dfrac{1-x}{1+x}$ in the equation

$$(f(x))^2\, f\left(\frac{1-x}{1+x}\right) = x^3 \qquad \text{...(i)}$$

We have

$$\left(f\left(\frac{1-x}{1+x}\right)\right)^2 f(x) = \left(\frac{1-x}{1+x}\right)^3 \qquad \text{... (ii)}$$

Solving (i) and (ii), we have

$$\left(\frac{x^3}{(f(x))^2}\right)^2 f(x) = \left(\frac{1-x}{1+x}\right)^3$$

$$\Rightarrow \qquad \frac{(f(x))^3}{x^6} = \left(\frac{1+x}{1-x}\right)^3$$

$$\Rightarrow \qquad f(x) = x^2\left(\frac{1+x}{1-x}\right)$$

$$f(-2) = 4\left(\frac{-1}{3}\right) = -\frac{4}{3}$$

$$\{f(-2)\} = -\frac{4}{3} - \left[-\frac{4}{3}\right] = -\frac{4}{3} + 2 = \frac{2}{3}.$$

Example 62: Let $f(x)$ be a function such that $f(x-1) + f(x+1) = \sqrt{2}f(x)$ for all $x \in \mathbf{R}$. If $f(3) = 5$ then $\sum_{r=0}^{10} f(3+8r)$ is equal to

(a) 50 (b) 55
(c) 0 (d) 10

Ans. (b)

Solution: The given equation is

$$f(x-1) + f(x+1) = \sqrt{2}f(x) \qquad \text{... (i)}$$

Replace x by $x-1$ in (i)

$$f(x-2) + f(x) = \sqrt{2}f(x-1) \qquad \text{... (ii)}$$

Replace x by $x+1$ in (ii)

$$f(x) + f(x+2) = \sqrt{2}f(x+1) \qquad \text{... (iii)}$$

Adding (ii) and (iii), we have

$$f(x-2) + f(x+2) + 2f(x) = \sqrt{2}\,(f(x-1) + f(x+1))$$

$$= \sqrt{2}\ \sqrt{2}f(x) = 2f(x)$$

$\Rightarrow \qquad f(x-2) = -f(x+2)$

Replacing x by $x+2$, we have

$$f(x) = -f(x+4) = -(f(x+8)) = f(x+8)$$

$$\sum_{r=0}^{10} f(3+8r) = 11 \times f(3) = 11 \times 5 = 55.$$

Example 63:

$$\text{Let } f(\theta) = \begin{vmatrix} \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \end{vmatrix}$$

then f is

(a) a non periodic function
(b) periodic with period π
(c) periodic with period $\pi/2$
(d) odd function

Ans. (c)

Solution:

$$f(\theta) = \begin{vmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1+\sin^2\theta & \cos^2\theta & 4\sin 2\theta \end{vmatrix}$$

$$(R_1 \to R_1 - R_3,\ R_2 \to R_2 - R_3)$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1+\sin^2\theta + 4\sin 4\theta & \cos^2\theta & 4\sin 4\theta \end{vmatrix} \qquad (C_1 \to C_1 + C_3)$$

$$= -(\cos^2\theta + 1 + \sin^2\theta + 4\sin 4\theta)$$

$$= -2\,(1 + 2\sin 4\theta)$$

which is periodic function with period $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$.

Worked Out Examples - Reasoning Type

Example 64: Let **R** be the real line. Consider the following subsets of the plane $\mathbf{R} \times \mathbf{R}$:

$$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$$
$$T = \{(x, y) : x - y \text{ is an integer}\}$$

Statement-1 : T is an equivalence relation on R but S is not an equivalence relation on R.
Statement-2 : S is neither reflexive nor symmetric but T is reflexive, symmetric and transitive.
Ans. (a)

Solution: Since $x \neq x + 1$, $(x, x) \notin S$, so S is not reflexive. Next $x, y \in S$, $\Rightarrow y = x + 1 \Rightarrow x = y - 1 \Rightarrow (y, x) \notin S$, so is not symmetric.
Since $x - x = 0$ is an integer $(x, x) \in T\ \forall\ x \in T$
$\Rightarrow$ T is reflexive.
Again $(x, y) \in T \Rightarrow x - y$ is an integer
$\Rightarrow$ $y - x$ is also an integer $\Rightarrow (y, x) \in T$
So T is symmetric.
Also $(x, y) \in T$, $(y, z) \in T$.
$\Rightarrow$ $x - y$ and $y - z$ are integers
$\Rightarrow$ $x - z = (x - y) - (y - z)$ is also an integer
$\Rightarrow$ $(x, z) \in T$
So T is Transitive.
Which shows that statement-2 is true and hence statement-1 is also true.

Example 65: Consider the following relations.
$R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$

$$S = \left\{\left(\frac{m}{n}, \frac{p}{q}\right)\right\}\ m, n, p, q, \text{ are integer such that } n \cdot q \neq 0 \text{ and } qm = pn\}$$

Statement-1: S is an equivalence relation but R is not an equivalence relation.
Statement-2: R and S both are symmetric.
Ans. (c)

Solution: Since $(0, 1) \in R$ but $(1, 0) \notin R$, R is not symmetric and hence is not an equivalence relation so statement-2 is false.
Next, For the relation S, $qm = pn \Rightarrow \dfrac{m}{n} = \dfrac{p}{q}$

Thus $\left(\dfrac{m}{n}, \dfrac{p}{q}\right) \in S \Rightarrow \dfrac{m}{n} = \dfrac{p}{q}$ which shows that S is reflexive and symmetric

Again, $\left(\dfrac{m}{n}, \dfrac{p}{q}\right) \in S$ and $\left(\dfrac{p}{q}, \dfrac{r}{s}\right) \in S$

$$\Rightarrow \quad \frac{m}{n} = \frac{p}{q} = \frac{r}{s} \quad \Rightarrow \left(\frac{m}{n}, \frac{r}{s} \in S\right)$$

Thus S is transitive and hence S is an equivalence relation. So, statement 1 is true.

Example 66: Let R be a relation on the set N of natural numbers defined by $n\,Rm \Leftrightarrow n$ is a factor of m (*i.e.* $n \mid m$).
Statement-1: R is not an equivalence relation
Statement-2: R is not symmetric
Ans. (a)

Solution: Statement-2 is true as $2 \mid 6 \Rightarrow 2\mathbf{R}6$ but 6 does not divide 2 so R is not symmetric $\Rightarrow R$ is not an equivalence relation and the statement-1 is also true.

Example 67: Let $A = \{1, 2, 3\}$ and $B = \{3, 8\}$
Statement-1: $(A \cup B) \times (A \cap B) = \{(1, 3), (2, 3), (3, 3), (8, 3)\}$
Statement-2: $(A \times B) \cap (B \times A) = \{(3, 3)\}$
Ans. (b)

Solution: $A \cup B = \{1, 2, 3, 8\}$, $A \cap B = \{3\}$
$\Rightarrow$ $(A \cup B) \times (A \cap B) = \{(1, 3), (2, 3), (3, 3), (8, 3)\}$
$\Rightarrow$ Statement-1 is True.
$(x, y) \in (A \times B) \cap (B \times A)$
$\Rightarrow$ $(x, y) \in A \times B$ and $(x, y) \in B \times A$
$\Rightarrow$ $x \in A \cap B$, $y \in A \cap B$
$\Rightarrow$ $\{(3, 3)\} = (A \times B) \cap (B \times A) \Rightarrow$ Statement-2 is also true but is not a correct explanation for statement-1.

Example 68: Statement-1: The number of bijective functions from the set A containing 100 elements to itself is 2^{100}.
Statement-2: The total number of bijections from a set containing n elements to itself is $n!$
Ans. (d)

Solution: Statement-2 is true and so, statement-1 is False.

Example 69: Statement-1: $f : R \to R$ is a function defined by $f(x) = 5x + 3$. If $g = f^{-1}$, then $g(x) = \dfrac{x-3}{5}$.
Statement-2: If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then fog is the identity function on A.
Ans. (c)

Solution: Let $y = 5x + 3 \Rightarrow x = \dfrac{y-3}{5}$.

$\Rightarrow$ $g(x) = \dfrac{x-3}{5}$ is the inverse of f, so statement-1 is True.

Statement-2 is false because $g : B \to A$ and $f : A \to B$

$\Rightarrow$ $fog : B \to B$ and $g = f^{-1} \Rightarrow fog$ is an identity function on B.

Example 70: Let X and Y be two sets.

Statement-1: $X \cap (Y \cup X)' = \phi$

Statement-2: If $X \cup Y$ has m elements and $X \cap Y$ has n elements then symmetric difference $X \Delta Y$ has $m - n$ elements

Ans. (b)

Solution: $X \cap (Y \cup X)' = X \cap (Y' \cap X') = X \cap X' \cap Y' = \phi$.

$\Rightarrow$ Statement-1 is True.

$X \Delta Y = (X \sim Y) \cup (Y \sim X) = (X \cup Y) \sim (X \cap Y)$

$\Rightarrow$ number of elements in $X \Delta Y = m - n$.

$\Rightarrow$ Statement-2 is True but does explain statement-1.

Example 71: Let f be a function defined by

$f(x) = (x - 1)^2 + 1, (x \geq 1)$

Statement-1: The set $\{x : f(x) = f^{-1}(x)\} = \{1, 2\}$

Statement-2: f is a bijection and $f^{-1}(x) = 1 + \sqrt{x-1}$, $x \geq 1$

Ans. (a)

Solution: Let $y = f(x) = (x - 1)^2 + 1$

$\Rightarrow$ $y - 1 = (x - 1)^2 \Rightarrow x = 1 + \sqrt{y-1}$, $y \geq 1$

Thus $f^{-1}(x) = 1 + \sqrt{x-1}$, $x \geq 1$. So statement-2 is true.

Now $f(x) = f^{-1}(x)$

$\Rightarrow$ $(x - 1)^2 = \sqrt{x-1}$

$\Rightarrow$ $\sqrt{x-1}\,[(x-1)^{3/2} - 1] = 0$

$\Rightarrow$ $x = 1, 2$.

So statement-1 is true and statement-2 is a correct explanation for statement-1.

Example 72: Let R be the set of real numbers

Statement-1: $A = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y + x$ is an integer$\}$ is an equivalence relation on $\mathbf{R}$.

Statment-2: $B = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y = \alpha x$ for some rational number $\alpha\}$ is not equivalence relation on $\mathbf{R}$.

Ans. (d)

Solution: A is neither reflexive nor transitive as $x + x$ may not be integer $\forall\, x \in \mathbf{R}$ and if $x + y$ and $y + z$ are integers, $x + z$ may not be an integer for $x, y, z \in \mathbf{R}$.

So statement-1 is false.

Statememt-2 is true as B is not symmetric, because $(\sqrt{3}, 0) \in B$ as $0 = \sqrt{3} \times 0$ for $\alpha = 0$, but $(0, \sqrt{3}) \notin B$.

Example 73: Consider the following relation R on the set of real square matrices of order 3.

$R = \{ (A, B): A = P^{-1}BP$ for some invertible matrix $P\}$

Statement-1: R is an equivalence relation.

Statement-2: For any two invertible 3×3 matrices M and N, $(MN)^{-1} = N^{-1}M^{-1}$.

Ans. (b)

Solution: Statement-2 in true (See Text.)

In statement-1, $A = I^{-1} A I$

for all real square matrices A of order 3.

$\Rightarrow$ $(A, A) \in R \Rightarrow R$ is reflexive,

Next, let $(A, B) \in R$

$\Rightarrow$ $\exists$ a invertible matrix P of order 3.

such that $A = P^{-1} B P$

$\Rightarrow$ $B = P A P^{-1} = (P^{-1})^{-1} A (P^{-1})$

$\Rightarrow$ R in symmetric

If Now $(A, B) \in R$ and $(B, C) \in R$

Then $\exists$ invertible matrices P and Q

of order 3 such that

$A = P^{-1} B P$ and $B = Q^{-1} C Q$.

$\Rightarrow$ $A = P^{-1} Q^{-1} C Q P = (QP)^{-1} C QP$ (From statement-2)

$\Rightarrow$ $(A, C) \in R$ and thus R in transitive. Hence R is an equivalence relation and the statement-1 in also true but statement-2 is not a correct explanation for it.

Worked Out Examples - Level 2

Example 74: From 50 students taking examinations in Mathematics, Physics and Chemistry, 37 passed Mathematics, 24 Physics and 43 Chemistry. At most 19 passed Mathematics and Physics, at most 29 Mathematics and Chemistry and at most 20 Physics and Chemistry. The largest possible number that could have passed all three exams is

(a) 10 (b) 12

(c) 9 (d) none of these.

Ans. (d)

Solution: The given conditions can be expressed as

$n(M \cup P \cup C) = 50$, $n(M) = 37$, $n(P) = 24$, $n(C) = 43$, $n(M \cap P) \leq 19$,

$n(M \cap C) \leq 29$ and $n(P \cap C) \leq 20$,

$n(M \cup P \cup C) = n(M) + n(P) + n(C) - n(M \cap P) - n(M \cap C) - n(P \cap C) + n(M \cap P \cap C)$

$\Rightarrow$ $50 = 37 + 24 + 43 - n(M \cap P) - n(P \cap C)$

$$- n(M \cap C) + n(M \cap P \cap C)$$

$$\Rightarrow \quad n(M \cap P \cap C) \le n(M \cap P) + n(M \cap C) + n(P \cap C) - 54.$$

Therefore, the number of students that could have passed all three exams is at most
$19 + 29 + 20 - 54 = 14$.

Example 75: Suppose $A_1, A_2, \ldots, A_{30}$ are thirty sets each having 5 elements and $B_1, B_2 \ldots, B_n$ are n sets each with 3 elements, let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$ and each element of S belongs to exactly 10 of the A_i's and exactly 9 of the B_j's. Then n is equal to

(a) 15 (b) 3
(c) 45 (d) none of these

Ans. (c)

Solution: $S = \bigcup_{i=1}^{30} A_i$, so $n(S) = \frac{1}{10}(5 \times 30) = 15$ (since element in the union S belongs to exactly 10 of the A_i's).
Again $S = \bigcup_{i=1}^{n} B_i$ so

$$n(S) = 1/9(3 \times n) = n/3 = 15 \Rightarrow n = 45$$

Example 76: Let $R = \{(3,3), (6,6), (9,9)\ (12,12), (6,12), (3,9), (3,12), (3,6)\}$ be a relation on the set A = $\{3, 6, 9, 12\}$.
The relation is

(a) an equivalence relation.
(b) reflexive and symmetric only.
(c) reflexive and transitive only
(d) reflexive only.

Ans. (c)

Solution: R is reflexive as
$(3, 3), (6, 6), (9, 9), (12, 12) \in R$.
R is not symmetric as $(6, 12) \in R$ but $(12, 6) \notin R$.
R is transitive as the only pair which needs verification is $(3, 6)$ and $(6, 12) \in R \Rightarrow (3, 12) \in R$.

Example 77: Let $R = \{(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)\}$ be a relation on the set A = $\{1, 2, 3, 4\}$, the relation R is

(a) not symmetric (b) transitive
(c) a function (d) reflexive

Ans. (a)

Solution: R is not symmetric as $(2, 3) \in R$ but $(3, 2) \notin R$. It is not transitive as $(1, 3), (3, 1) \in R$. but $(1, 1) \notin$ R so is not reflexive also.
Again as $(2, 4)$ and $(2, 3) \in R$, it is not a function.

Example 78: Let **I** be the set of integers, **N** the set of non-negative integers; Np the set of non-positive integers ; E is the set of even integers and P is set of prime numbers. Then

(a) $\mathbf{N} \cap Np = \phi$ (b) $\mathbf{I} - \mathbf{N} = Np$
(c) $\mathbf{N}\, \Delta\, Np = I - \{0\}$ (d) $E \cap P = \phi$.

Ans. (c)

Solution: $N \cap Np = \{0\}$,

$$\mathbf{I} - N = \{\ldots, -2, -1\}$$
$$N\, \Delta\, Np = \{N - Np) \cup (Np - N)$$
$$= \{1, 2, 3 --\} \cup \{\ldots, -3, -2, -1\}$$
$$= \mathbf{I} \sim \{0\}$$

and $E \cap P = \{2\}$.

Example 79: If $n(A) = n$ then $n\{(x, y, z); x, y, z \in A, x \ne y, y \ne z, z \ne x\} =$

(a) n^3 (b) $n(n-1)^2$
(c) $n^2(n-2)$ (d) $n^3 - 3n^2 + 2n$

Ans. (d)

Solution: There are n choices for the first coordinate, $n - 1$ choices for second coordinate and $n - 2$ choices for the third coordinate, hence $n(\{(x, y, z); x, y, z \in A, x \ne y \ne z\}) = n(n-1)(n-2) = n^3 - 3n^2 + 2n$.

Example 80: If $A = \{x : x \in \mathbf{I}, -2 \le x \le 2\}$, $B = \{x \in \mathbf{I}, 0 \le x \le 3\}$, $C = \{x: x \in \mathbf{N}, 1 \le x \le 2\}$ and $D = \{x, y) \in \mathbf{N} \times \mathbf{N}; x + y = 8\}$. Then

(a) $n(A \cup (B \cup C) = 5$ (b) $n(D) = 6$
(c) $n(B \cup C) = 5$ (d) none of these

Ans. (d)

Solution: $A = \{-2, -1, 0, 1, 2\}$, $B = \{0, 1, 2, 3\}$, $C = \{1, 2\}$

so $B \cup C = \{0, 1, 2, 3\}$,
$A \cup (B \cup C) = \{-2, -1, 0, 1, 2, 3\}$
so $n(A \cup (B \cup C) = 6$, $n(B \cup C) = 4$
and $D = \{(1, 7), (2, 6), (3, 5), (4, 4), (5,3), (6, 2), (7, 1)\}$ so
$n(D) = 7$.

Example 81: If $A = \{4^n - 3n - 1 | n \in \mathbf{N}\}$ and $B = \{9n - 9: n \in \mathbf{N}\}$, then $A \cup B$ is equal to

(a) B (b) A
(c) **N** (d) none of these

Ans. (a)

Solution: It can be shown by induction that $9 \mid 4^n - 3n - 1$ for every $n \in \mathbf{N}$. Thus $A \subseteq B$. Clearly $A \ne B$ as $27 \in B$ but $27 \notin A$. Thus $A \cup B = B$.

Example 82: A and B are two sets having 3 and 4 elements respectively and having 2 elements in common. The number of relations which can be defined from A to B is

(a) 2^5 (b) $2^{10} - 1$
(c) 2^{12} (d) none of these.

Ans. (c)

Solution: The number of elements in $A \times B$ is 12. Hence the number of subsets of $A \times B$ is 2^{12}, which includes the empty set

Example 83: If R and S are two symmetric relations then

(a) $R\ o\ S$ is a symmetric relation
(b) $S\ o\ R$ is a symmetric relation
(c) $R\ o\ S^{-1}$ is a symmetric relation
(d) $R\ o\ S$ is a symmetric relation if and only if $RoS = SoR$.

Ans. (d)

Solution: Since R and S are symmetric relations so $R^{-1} = R$ and $S^{-1} = S$. But $(R\ o\ S)^{-1} = S^{-1}\ o\ R^{-1} = S\ o\ R$. Thus RoS is symmetric if and only if $RoS = SoR$.

Example 84: Let A be the set of all determinants of order 3 with entries 0 or 1 only, B the subset of A consisting of all determinants with value 1, and C the subset consisting of all determinants with value –1. Then if $n(B)$ and $n(C)$ denote the number of elements in B and C, respectively, we have

(a) $C = \phi$ (b) $n(B) = n(C)$
(c) $A = B \cup C$ (d) $n(B) = 2n(C)$

Ans. (b)

Solution: C cannot be the empty set because, for instance,

$$-1 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \in C. \text{ We also have } \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2,$$

so $A \neq B \cup C$. In general, the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{11}a_{22}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}, \end{matrix}$$

with the a's being 0 or 1, equals 1 only if $a_{11}a_{22}a_{33} = 1$ and the remaining terms are zero; if $a_{12}\,a_{23}\,a_{31} = 1$ and the remaining terms are zero; or if $a_{13}\,a_{21}\,a_{32} = 1$ and the remaining terms are zero. Since there are three similar relations for determinants that equal – 1, we must have $n(B) = n(C)$.

Example 85: The domain of the function

$$f(x) = \log_2\left(-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1\right) \text{ is}$$

(a) $0 < x < 1$ (b) $0 < x \le 1$
(c) $x \ge 1$ (d) $x > 1$

Ans. (a)

Solution: For f to be defined we must have $\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right) < -1 \Leftrightarrow 1 + \frac{1}{\sqrt[4]{x}} > (2^{-1})^{-1} = 2$ which is possible if and only if $\frac{1}{\sqrt[4]{x}} > 1$ i.e., $0 < x < 1$.

Hence the domain of the given function is $\{x : 0 < x < 1\}$.

Example 86: The domain of definition of

$$f(x) = \sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right)} \times \frac{1}{x^2-36} \text{ is}$$

(a) $\{x : x < 0, x \neq -6\}$
(b) $\{x : x > 0, x \neq 1, x \neq 6\}$
(c) $\{x : x > 1, x \neq 6\}$
(d) $\{x : x \ge 1, x \neq 6\}$

Ans. (c)

Solution: For $\sqrt{\log_{0\cdot4}\left(\frac{x-1}{x+5}\right)}$ to be defined, we must have $0< \frac{x-1}{x+5} < 1$, which is true if $x > 1$. Morever, $\frac{1}{x^2-36}$ is defined for $x \neq \pm 6$. Hence the domain of f is $\{x : x > 1, x \neq 6\}$.

Example 87: The set of all x for which $f(x) = \log_{\frac{x-2}{x+3}} 2$ and $g(x) = \frac{1}{\sqrt{x^2-9}}$ are both not defined is

(a) $(-3, 2)$ (b) $[-3, 2)$
(c) $(-3, 2]$ (d) $[-3, 2]$

Ans. (d)

Solution: The function f is not defined for $-3 \le x \le 2$ and g is not defined for those x for which $x^2 - 9 \le 0$ i.e., $x \in [-3, 3]$. Thus f and g are not defined on $[-3, 2]$.

Example 88: If $f(x)$ is a polynomial satisfying $f(x)\,f(1/x) = f(x) + f(1/x)$ and $f(3) = 28$, then $f(4)$ is given by

(a) 63 (b) 65
(c) 67 (d) 68

Ans. (b)

Solution: Any polynomial satisfying the functional equation $f(x) \cdot f(1/x) = f(x) + f(1/x)$ is of the form $x^n + 1$ or $-x^n + 1$. If $28 = f(3) = -3^n + 1$ then $3^n = -27$ which is not possible for any n. Hence $28 = f(3) = 3^n + 1 \Rightarrow 3^n = 27 \Rightarrow n = 3$. Thus $f(x) = x^3 + 1$, so $f(4) = 4^3 + 1 = 65$.

Example 89: Part of the domain of the function

$f(x) = \sqrt{\frac{\cos x - 1/2}{6+35x-6x^2}}$ lying in the interval $[-1, 6]$ is

(a) $[-1/6, \pi/3] \cup [5\pi/3, 6]$
(b) $(-1/6, \pi/3] \cup [5\pi/3, 6)$
(c) $(-1/6, 6)$
(d) none of these

Ans. (a)

Solution: The function f is meaningful only if $\cos x - 1/2 \ge 0$, $6 + 35x - 6x^2 > 0$ or $\cos x - 1/2 \le 0$, $6 + 35x - 6x^2 < 0$ i.e., $\cos x \ge 1/2$, $(6 - x)(1 + 6x) > 0$ or $\cos x \le 1/2$, $(6 - x)(1 + 6x) < 0$. These inequalities are satisfied if $x \in (-1/6, \pi/3] \cup [5\pi/3, 6)$.

Example 90: Let f: $\mathbf{R} \to \mathbf{R}$ be a function defined by $f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$. Then

(a) f is both one-one and onto
(b) f is one-one but not onto

(c) f is onto but not one-one
(d) f is neither one-one nor onto

Ans. (d)

◎ **Solution:** f is not one-one as $f(0) = 0$ and $f(-1) = 0$. f is also not onto as for $y = 1$ there is no $x \in \mathbf{R}$ such that $f(x) = 1$. If there is such a $x \in \mathbf{R}$ then $e^{|x|} - e^{-x} = e^x + e^{-x}$, clearly $x \neq 0$. For $x > 0$, this equation gives $-e^{-x} = e^{-x}$ which is not possible. For $x < 0$, the above equation gives $e^x = -e^{-x}$ which is also not possible.

◉ **Example 91:** Let $f(x) = x^2$ and $g(x) = 2^x$ then the solution set of $fog\ (x) = g\ o f(x)$ is

(a) $\mathbf{R}$ (b) $\{0\}$
(c) $\{0, 2\}$ (d) none of these

Ans. (c)

◎ **Solution:** $fog\ (x) = f(g(x)) = f(2^x) = (2^x)^2 = 2^{2x}$ and $gof\ (x) = g\ (f(x)) = g\ (x^2) = 2^{x^2}$. Thus the solution of $2^{x^2} = 2^{2x}$ is given by $x^2 = 2x$ which is $x = 0, 2$.

◉ **Example 92:** A function $f: \mathbf{R} \to \mathbf{R}$ satisfies the equation $f(x) f(y) - f(xy) = x + y$ for all $x, y \in \mathbf{R}$ and $f(1) > 0$, then

(a) $f(x) = x + 1/2$ (b) $f(x) = (1/2)x + 1$
(c) $f(x) = (1/2)x - 1$ (d) $f(x) = x + 1$

Ans. (d)

◎ **Solution:** Taking $x = y = 1$, we get

$$f(1) f(1) - f(1) = 1 + 1 \Rightarrow f(1)^2 - f(1) - 2 = 0$$

$\Rightarrow (f(1) - 2)(f(1) + 1) = 0 \Rightarrow f(1) = 2\ (\because f(1) > 0)$

Taking $y = 1$, we get

$$f(x) f(1) - f(x) = x + 1 \Rightarrow 2f(x) - f(x) = x + 1$$

$$\Rightarrow \quad f(x) = x + 1.$$

◉ **Example 93:** If $f: [1, \infty) \to [2, \infty)$ is given by $f(x) = x + 1/x$ then $f^{-1}(x)$ equals

(a) $\dfrac{x + \sqrt{x^2 - 4}}{2}$ (b) $\dfrac{x}{1 + x^2}$

(c) $\dfrac{x - \sqrt{x^2 - 4}}{2}$ (d) $1 + \sqrt{x^2 - 4}$

Ans. (a)

◎ **Solution:** $y = x + 1/x \Rightarrow x^2 - xy + 1 = 0$

$$\Rightarrow \quad x = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

Since $x \in [1, \infty)$ so $x = \dfrac{y + \sqrt{y^2 - 4}}{2}$.

Hence $f^{-1}(x) = \dfrac{x + \sqrt{x^2 - 4}}{2}$.

◉ **Example 94:** Consider the function $f = \{(x, \sin x) | -\infty < x < \infty\}$. Let $A = \left[0, \dfrac{\pi}{6}\right]$ and $B = \left[0, \dfrac{\pi}{2}\right]$ then

(a) $(A \cap B) = f(A) \cap f(B)$ (b) $f(A \cap B) = [0, 1]$

(c) $f(A) \cap f(B) = [0, 1]$ (d) $f(A) \cup f(B) = \left[0, \dfrac{1}{2}\right]$

Ans. (a)

◎ **Solution:** $f(x) = \sin x$ is an increasing function on $[0, \pi/2]$ so $f(A) = \left[0, \sin\dfrac{\pi}{6}\right] = \left[0, \dfrac{1}{2}\right]$ and $f(B) = [0, 1]$. Thus $f(A) \cap f(B) = \left[0, \dfrac{1}{2}\right]$. Also $A \cap B = \left[0, \dfrac{\pi}{6}\right]$, so $f(A \cap B) = \left[0, \sin\dfrac{\pi}{6}\right] = \left[0, \dfrac{1}{2}\right]$. Thus $f(A) \cap f(B) = \left[0, \dfrac{1}{2}\right] = f(A \cap B)$, Also $f(A) \cup f(B) = [0, 1]$.

◉ **Example 95:** Let $f(x)$ be a polynomial of even degree satisfying $f(2x)\left(1 - f\left(\dfrac{1}{2x}\right)\right) + f\left(16x^2 y\right) = f(-2) - f(4xy)$ for all $x, y \in \mathbf{R} \sim \{0\}$ and $f(4) = -255$, $f(0) = 1$. Then the value of $\left|\dfrac{f(2)+1}{2}\right|$ is

(a) 4 (b) 5
(c) 7 (d) 6

Ans. (c)

◎ **Solution:** Replacing y by $\dfrac{1}{8x^2}$ in the given functional equation, we obtain

$$f(2x)\left(1 - f\left(\frac{1}{2x}\right)\right) + f(2) = f(-2) - f\left(\frac{1}{2x}\right).$$

Since f is an even function so $f(2) = f(-2)$,

$$\text{so} \quad f(2x) - f(2x) f\left(\frac{1}{2x}\right) = -f\left(\frac{1}{2x}\right)$$

$$\Rightarrow \quad f(2x) + f\left(\frac{1}{2x}\right) = f(2x)\, f\left(\frac{1}{2x}\right)$$

Replacing x by $\dfrac{x}{2}$, we have

$$f(x) + f\left(\frac{1}{x}\right) = f(x)\, f\left(\frac{1}{x}\right)$$

Since f is a polynomial, so $f(x) = \pm x^n + 1$ (see Example 88) But $-255 = f(4) = \pm 4^n + 1$. Only negative sign is possible, thus $4^n = 256 \Rightarrow n = 4$. i.e. $f(x) = -x^4 + 1$. $f(2) = -2^4 + 1 = -15$

$$\left|\frac{f(2)+1}{2}\right| = \left|-\frac{14}{2}\right| = 7.$$