Solutions

Exercise - Concept-based Questions

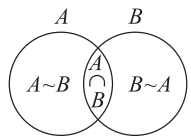
1. $n(A \cup B') = n(A) + n(B') - n(A \cap B')$
$= n(A) + n(U) - n(B) - (n(A) - n(A \cap B))$
$= n(U) - n(B) + n(A \cap B)$
$= 12 - 6 + 2 = 8.$
2. If $a = -1$ than a is not related to a. So R is not reflexive. If $a = -1$, $b = 2$ then $|a| \le b$ but $|b| > a$, so R is not symmetric. If $|a| \le b$ and $|b| \le c$ then $|a| \le b \le |b| \le c$ so $|a| \le c$, hence a is related to c. Thus R is transitive. If $|a| \le b$, and $|b| \le a$ then $a \le |a| \le b \le |b| \le a$. So $a = b$, R is anti symmetric.
3. $f(g(x)) = g(f(x))$ for all $x \in \mathbf{R} \Leftrightarrow f(cx + d) = g(ax + b)$
$\Leftrightarrow a(cx + d) + b = c(ax + b) + d \Leftrightarrow ad + b = cb + d$
$\Leftrightarrow f(d) = g(b)$
4. For $f(x)$ to be defined, we have $|x| > x$. Since $|x| \ge x \ \forall \ x \in \mathbf{R}$ and $|x| = x$ for $x \in [0, \infty)$ so $|x| > x$ if $x \in \mathbf{R} = (-\infty, 0)$
5. The function is defined if $-1 \le 1 - 2x \le 1 \Leftrightarrow$
$-2 \le -2x \le 0$
$\Leftrightarrow 0 \le x \le 1.$
6. $f_1(x) = \dfrac{f(x) + f(-x)}{2} = \dfrac{1}{2}[\sin x - \cos x - \sin x - \cos x]$
$= -\cos x$
$f_2(x) = \dfrac{f(x) - f(-x)}{2} = \dfrac{1}{2}[\sin x - \cos x + \sin x + \cos x]$
$= \sin x$
7. $-1 \le \sin x \le 1 \Leftrightarrow -1 \le -\sin x \le 1 \Leftrightarrow 0 \le 1 - \sin x \le 2$
8. $f(-x) = (-x)^2 \log \dfrac{1+x}{1-x} = x^2 \log \left(\dfrac{1-x}{1+x}\right)^{-1} =$
$-x^2 \log \dfrac{1-x}{1+x} = -f(x).$

Exercise - Level 1

9. $((A \sim B) \cup (B \sim A)) \cap A$
$= ((A \sim B) \cap A) \cup (B \sim A) \cap A$
$= (A \sim B) \cup \phi = (A \sim B) = \{3, 5\}$
10. $(A \cap B) \cap C = \{2 \times 3 \times 5\ x : x \in N\}$
$= \{30x : x \in \mathbf{N}\}$
11. $X \subset Y \Rightarrow X \cap Y = X, X \cup Y = Y$
$Y \subset X \Rightarrow X \cap Y = Y, X \cup Y = X.$
So $X \cap Y = X \cup Y \Rightarrow X = Y$
12. If $X \subsetneq Y$, let $y \in Y$ and $y \notin X$ then either $y \notin A$. or $y \in A$
So if $y \in A$, then $y \in A \cap Y$ but $y \notin A \cap X$ and thus $A \cap X \ne A \cap Y$.
If $y \notin A$, then $y \in A \cup Y$ but $y \notin A \cup X$ and thus $A \cup X \ne A \cup Y$
So $X \subseteq Y$, similarly $Y \subseteq X$.
$\Rightarrow \quad X = Y.$
13. $A = \{7, 8\}$, $B = \{1, 2, 3, ... 9\}$
$C = \{1, 2, 4, 7, 8, 14, 16, ...\}$
$B \cap C = \{1, 2, 4, 7, 8\}$
$A \cup (B \cap C) = \{1, 2, 4, 7, 8\}$
14. $A = \{M, A, T, H, E, I, C, S\}$
$B = \{S, T, A, I, C\}$
$B \sim A = \phi$, So $A \Delta B = (A \sim B) \cup (B \sim A)$
$= A \sim B.$
Verify (a) and (b) are not correct.

15. $n(A \sim B) = n(A) - n(A \cap B)$
$\Rightarrow \quad 15 = n(A) - 16 \Rightarrow n(A) = 31$
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$\Rightarrow \quad 36 = 31 + n(B) - 16$
$\Rightarrow \quad n(B) = 21.$

16. $A \cup B, A \sim B, B \sim A$



$(A \cup B) \sim (A \sim B) = B$
$(A \cup B) \sim (B \sim A) = A$
$A \,\Delta\, B = (A \sim B) \cup (B \sim A)$
$(A \cup B) \sim (A \,\Delta\, B) = A \cap B$
and $(A \sim B) \sim A = \phi$
Thus, the required no is 8.

17. $n(A \sim B) = n(A) - n(A \cap B)$
$= n(B) - n(A \cap B) = n(B \sim A)$

18. $A \cup B = A \,\Delta\, B \cup (A \cap B)$
So $A \cup B = A \,\Delta\, B \Rightarrow A \cap B = \phi$.

19. $n(P) = 60$, $n(M) = 58$, $n(P \cup M) = 100$, $N(P \cap M)$
$= n(P) + M(M) - n(P \cup M)$

20. $R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5)\}$

21. $(1, 2) \in R$ but $(1, 2) \notin R_1$ or R_2 or R_3 so $R \neq R_1$ or R_2 or R_3.

22. L_1 is not $\perp L_1$, so R is not reflexive
$L_1 \perp L_2 \Rightarrow L_2 \perp L_1 \Rightarrow R$ is symmetric.
$L_1 \perp L_2$ and $L_2 \perp L_3 \Rightarrow L_1 \parallel L_3 \Rightarrow R$ is not transitive.

23. See the definition of function

24. $f(x) = f^{-1}(x) \Rightarrow f(f(x)) = x.$
$\Rightarrow \quad [(x + 1)^2 - 1 + 1]^2 - 1 = x.$
$\Rightarrow \quad (x + 1)^2 = x + 1 \Rightarrow x = 0$ or $x = -1$
So $S = \{0, -1\}$.

25. $A \times A$ has $n \times n = n^2$ elements and any subset of $A \times A$ is a relation on A. The number of such subsets is 2^{n^2}.

26. $-1 \leq \sin x \leq 1 \Rightarrow 1 \leq f(x) \leq 5 \Rightarrow f(x)$ is not onto, $f(x) = f(x + 2\pi) \Rightarrow f$ is not one-one and f is not bijective.

27. In (a) $a \to 2a$ and $a \to -2a$, so it is not a function
In (b) for each $x \in R$ there is unique $|x| = y \in R$, so it is a function.

28. $f(x) = x^4 + 2 = y \Rightarrow x = (y - 2)^{1/4}$.
$\Rightarrow \quad f^{-1}(x) = (x - 2)^{1/4}$
$\Rightarrow \quad f^{-1}(83) = (81)^{1/4} = \pm 3$
and $f^{-1}(-2) = (-4)^{1/4} = \phi$.

29. R is reflexive if $(x, x). \in R$ for all $x \in \{1, 2, 3\}$
$\Rightarrow \quad (1, 1), (2, 2), (3, 3) \in R$
R is symmetric if $(1, 2), (2, 3) \in R$
$\Rightarrow \quad (2, 1), (3, 2) \in R$.
R is transitive if $(1, 2), (2, 3) \in R$
$\Rightarrow \quad (1, 3) \in R$ also $(3, 1) \in R$ as
R is symmetric.
So the total numbers of elements is 9.

30. $A \cap A = A \neq \phi$ if $A \neq \phi$, So R is not reflexive.
$A \cap B = \phi \quad \Rightarrow \quad B \cap A = \phi$
$ARB \quad \Rightarrow \quad BRA \quad \Rightarrow \quad R$ is symmetric
Note R is not transitive and not an equivalence relation.

31. $2f(2) - 3f(1/2) = 2^2 = 4$
$2f(1/2) - 3f(2) = (1/2)^2 = 1/4$
$\Rightarrow \quad 5(f(2) - f(1/2)) = 4 - 1/4 = 15/4$
$\Rightarrow \quad f(1/2) = f(2) - 3/4.$
So $\quad 2f(2) - 3[f(2) - 3/4] = 4$
$$\Rightarrow \quad f(2) = \frac{9}{4} - 4 = -\frac{7}{4}.$$

32. Applying $C_3 \to C_3 - (C_1 + C_2)$ we get $f(x) = 0$ for all x.

33. $a^{x-1} + a^{y-1} = 1$
$\Rightarrow \quad a^{y-1} = 1 - a^{x-1} \Rightarrow (y - 1)\log a = \log(1 - a^{x-1})$
R.H.S is defined if $1 - a^{x-1} > 0$
$\Rightarrow \quad a^{x-1} < 1 = a^{\circ} \Rightarrow x - 1 < 0$ as $a > 1$
$\Rightarrow \quad x < 1$ and the required domain is $-\infty < x < 1$.

34. If x is an integer, $f(x) = 0$
if x is not an integer $x - [x] = \dfrac{p}{q}$ where p and q are positive integers and $p < q$, so that $f(x) = \dfrac{p}{p+q} < \dfrac{1}{2}$
so $0 \leq f(x) < \dfrac{1}{2}$ and the required range is $[0, \dfrac{1}{2}]$

35. $2f(x^2) + 3f(1/x^2) = x^2 - 1$
$\Rightarrow \quad 2f(1/x^2) + 3f(x^2) = 1/x^2 - 1$
Subtracting, $f(x^2) - f(1/x^2) = 1/x^2 - x^2$
$\Rightarrow \quad f(x^2) = 1/x^2.$

36. $f(x + 3y, x - 3y) = 12xy = (x + 3y)^2 - (x - 3y)^2$
$\Rightarrow \quad f(x, y) = x^2 - y^2.$

37. Let $f(x) = y = 2^{x(x-1)}$
$\Rightarrow \quad \log y = x(x - 1) \log^2$
$\Rightarrow \quad x^2 - x - \log_2 y = 0$
$$\Rightarrow \quad x = \frac{1 \pm \sqrt{1 + 4\log_2 y}}{2}$$
$$\Rightarrow \quad x = f^{-1}(y) = \frac{1 \pm \sqrt{1 + 4\log_2 y}}{2}$$
$$\Rightarrow \quad f^{-1}(x) = \frac{1 + \sqrt{1 + 4\log_2 y}}{2} \text{ as } f^{-1}(x) \geq 1.$$

38. Number of one-one functions $= 5 \times 4 \times 3 = 60$.

39. If we take $f(x) = 1$, then $f(y) \neq 1$ as the function is one-one. So $f(x) \neq 1$
If $f(z) = 1$, then statements 2 and 3 both are true, so $f(z) \neq 1$ hence $f(y) = 1$ and $f^{-1}(1) = y$.
[**Note** $f(x) = 2, f(y) = 1, f(z) = 3$)]

40. $f(x + 4\pi) = f(x)$
$$\text{If } \frac{\sin n\,(x + 4\pi)}{\sin \dfrac{x + 4\pi}{n}} = \frac{\sin nx}{\sin \dfrac{x}{n}}$$
which holds if $n = 2$.

41. $(n, n) \in R$ only for $n = 1$

$m = n^2 \Rightarrow n = m^2$ unless $m = n = 1$

$m = n^2$ and $n = p^2 \not\Rightarrow m = p^2$ for $m, n, p \neq 1$

hence (a), (b), (c) none is true.

42. Let B, H, F be the sets of the three teams respectively

so $n(B) = 21, n(H) = 26, n(F) = 29,$

$n(H \cap B) = 14$ $n(H \cap F) = 15, n(F \cap B) = 12,$

$n(B \cap H \cap F) = 8$

and $n(B \cup H \cup F) = n(B) + n(H) + n(F) - n(H \cap B)$

$- n(H \cap F) - n(F \cap B) + n(B \cap H \cap F) = 21 + 26$

$+ 29 - 14 - 15 - 12 + 8 = 43$

43. An equilateral triangle cannot be obtuse angled or right angled but is an isosceles triangle.

So $E \subset I$ and $E \cap I = E$.

44. $A = \{3, 9, 27, 81, 243, 729\}, B = \{9, 81, 729, 6561\}$

$A \cup B = \{3, 9, 27, 81, 243, 729, 6561\}$

$A \cap B = \{9, 81, 729\}$

$A \sim B = \{3, 27, 243\}$

$B \sim A = \{6561\}$

$A \Delta B = (A \sim B) \cup (B \sim A) = \{3, 27, 243, 6561\}$

45. For $a \neq 0$, the range of f is **R**. f is differentiable function so f is one-one and only if f is monotonic.

$$f'(x) = a + \cos x$$

If $a > 1$, $f'(x) > 0$ i.e. f is increasing

If $a < -1$, $f'(x) < 0$ i.e. f is decreasing

Thus f is monotonic if $a \in \mathbf{R} \sim [-1, 1]$.

46. $f(x) = \sum_{k=1}^{n} 1 + [\sin kx] = n + [\sin x] + [\sin 2x] + \cdots + [\sin nx]$. If $kx \neq \frac{\pi}{2}$ for any $k = 1, 2 \ldots n$ then $0 < kx < \pi$ and $kx \neq \pi/2$ so $0 < \sin kx < 1$. Hence $[\sin kx] = 0$, $k = 1, 2 \ldots n$. i.e. $f(x) = n$. If $kx = \frac{\pi}{2}$ for some k then $x = \frac{\pi}{2k}$, hence $\sin x, \sin 2x, \ldots, \sin(k-1)x$ will lie between 0 and 1 so $[\sin jx] = 0$ $1 \leq j \leq k-1$; $\sin kx = 1$ so $f(x)$ can be $n + 1$ or n.

47. $(A \Delta B) \Delta C$ is disjoint union of $(A \sim B) \sim C$, $(B \sim C) \sim A$, $(C \sim A) \sim B$ and $A \cap B \cap C$.

Threfore, number of elements is $(A \Delta B) \Delta C$ is 10 + 15 + 20 +5 = 50.

48. $f^2(x) = fof(x) = f(f(x)) = f\left(\frac{x-3}{x+1}\right)$

$$= \frac{\frac{x-3}{x+1} - 3}{\frac{x-3}{x+1} + 1} = -\frac{x+3}{x-1}$$

$$f^3(x) = f\left(-\frac{x+3}{x-1}\right) = \frac{\frac{-x-3}{x-1} - 3}{\frac{-x-3}{x-1} + 1} = x.$$

Hence $f^{3k}(x) = x$

$$f^{2010}(2014) = f^{3.670}(2014) = 2014.$$

49. $(1, 2) \in R$, But $(2, 1) \notin R \Rightarrow R$ is not symmetric and hence not an equivalence relation.

$\Rightarrow$ Statement-1 is True. Staement-2 is False as $1 \to 1, 2, 3$.

50. Statement-2 is True. Because if A has m elements, then first elements can be mapped to n elements. For the 2nd element the choice is $n - 1$ and so on. So the total number of injective mappings is $n(n - 1)(n - 2) \ldots\ldots (n - m + 1) = n! / (n - m)!$ Which shows that statement-1 is False.

51. $f(-x) = -\sin x + \cos x \neq f(x)$ or $-f(x)$, $\Rightarrow f$ is neither odd nor even. So statement-1 is True

$g(-x) = \frac{-\sin x}{1 - \cos x} = -(g(x)) \Rightarrow g$ is an odd function

$\Rightarrow$ Statement-2 is also True but does not lead to statement-1.

52. Taking $y = 3$, $f(x) - f(3) = x - 3$

$\Rightarrow f(x) = x - 3 + f(3) = x - 3 + 2 = x - 1$

$\Rightarrow f(xy) = xy - 1 \Rightarrow$ Statement –1 is True

$$f(x) = \frac{x^2 + x + 1}{x^2 - x + 1} \Rightarrow f(1/x) = \frac{1 + x + x^2}{1 - x + x^2} = f(x)$$

and $f^{(2)} = 7/3 \Rightarrow$ Statement-2 is also True, but does not lead to statement-1.

53 . Statement-2 is true by definition of a bijective mapping, using which statement-1 is also true.

Exercise - Level 2

54. $-\pi/2 < x < \pi/2 \Rightarrow [x] = -2, -1, 0, 1$

$\Rightarrow f(x) = \cos[x]$

$= \cos(-2), \cos(-1), \cos(0), \cos 1$

$= \cos 2, \cos 1, 1 \cos 1$

55. $\frac{x-5}{x^2 - 10x + 24} > 0$ and $x + 5 > 0$

$\Rightarrow x > 5, x > 6 \Rightarrow x > 6 \Rightarrow x \in (6, \infty)$

or $x < 5, x > 4 \Rightarrow 4 < x < 5 \Rightarrow x \in (4, 5)$

56. Let $g(x) = 1 + \sqrt[3]{x} = y \Rightarrow x^{1/3} = y - 1$

so $f(g(x)) = 3 - 3x^{1/3} + x$

$\Rightarrow f(y) = 3 - 3(y - 1) + (y - 1)^3$

$= y^3 - 3y^2 + 5$

57. $f(-x) = \frac{-x}{e^x - 1} - \frac{x}{2} + 1$

$$= \frac{-xe^x}{1 - e^x} - \frac{x}{2} + 1$$

$$= \frac{x(e^x - 1 + 1)}{e^x - 1} - \frac{x}{2} + 1$$

$$= \frac{x}{e^x - 1} + \frac{x}{2} + 1 = f(x)$$

58. $(fog)\ (x) = f(g(x)) = f\left(\frac{x}{1-x}\right)$

$$= \frac{\frac{x}{1-x}}{\frac{x}{1-x} + 1} = \frac{x}{x + 1 - x} = x$$

$\Rightarrow (fog)^{-1}\ (x) = x$

59. $f(x) = \cot^{-1} \frac{2x}{1-x^2}$ is clearly one-one as

$f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$

Next, let $y = f(x) \Rightarrow \cot y = \frac{2x}{1-x^2}$

$$= \frac{2\tan(\theta/2)}{1 - \tan^2(\theta/2)} = \tan\theta$$

$\Rightarrow y = \pi/2 - \theta = \pi/2 - 2\tan^{-1}x$. Taking $x = \tan(\theta/2)$

so $y \in (0, \pi) \Rightarrow 0 < y < \pi$

$\Rightarrow -\pi/4 < \tan^{-1} x < \pi/4$

$\Rightarrow -1 < x < 1 \Rightarrow x \in (-1, 1)$

and thus f is onto.

60. As $-\sqrt{5^2 + 12^2} \le 5 \sin x + 12 \cos x \le \sqrt{5^2 + 12^2}$ for all x

$\Rightarrow 14 - 13 \le 5 \sin x + 2 \cos x \le 14 + 13$

$\Rightarrow 1 \le f(x) \le 27$

f is not one-one as it is periodic.

61. $fof\ (\cos x) = f\left(\frac{1 - \cos x}{1 + \cos x}\right) = f(\tan^2(x/2))$

$$= \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} = \cos x.$$

62. Since $f(x + T) = \sin \sqrt{x + T} \ne \sin\sqrt{x}$ for any value T.

$f(x)$ is not periodic.

63. Let $y = f(x) = \frac{x+2}{x-1}$, $y \in Y = \mathbf{R} - \{1\}$

$\Rightarrow x = \frac{y+2}{y-1}$, $x \in X = \mathbf{R} \sim \{1\}$

$\Rightarrow f$ is onto

Also $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \Rightarrow f$ is one-one.

64. $f(x + y) + f(x - y)$

$$= \frac{a^{x+y} + a^{-(x+y)}}{2} + \frac{a^{x-y} + a^{-(x-y)}}{2}$$

$$= \frac{a^x(a^y + a^{-y}) + a^{-x}(a^{-y} + a^y)}{2}$$

$$= \frac{\left(a^x + a^{-x}\right)\left(a^y + a^{-y}\right)}{2}$$

$= 2.\ f(x)\ f(y) \Rightarrow k = 2.$

65. $x - 3 \ge 0 \Rightarrow x \ge 3$ and $-1 \le \sqrt{x-3} \le 1$

$\Rightarrow x \le 4. \Rightarrow x \in [3, 4].$

66. $x[x] = \sqrt{|x|} \Rightarrow x^2 = x$ if x is an integer

$\Rightarrow x = 0, 1$

67. $f(x) = x\frac{a^x + a^{-x}}{a^x - a^{-x}} \Rightarrow f(-x) = f(x) \Rightarrow f$ is not odd.

$f(x) = \frac{a^x + x}{a^x - x} \Rightarrow f(-x) = \frac{1 - xa^x}{1 + xa^x} \ne -f(x)$

$\Rightarrow f$ is not odd

$f(x) = \frac{a^x - 1}{a^x + 1} \Rightarrow f(-x) = \frac{1 - a^x}{1 + a^x} = -f(x) \Rightarrow f$ is odd.

$f(x) = x \log_2 \sqrt{\left(x + \sqrt{x^2 + 1}\right)} \Rightarrow f(-x)$

$= -x \log \left(-x + \sqrt{x^2 + 1}\right) \ne f(-x).$

68. $g(x) = 1 + x - [x] > 0$ for all x

$\Rightarrow (fog)\ (x) = f(g(x)) = 1.$

69. The function $f(x) = \log_2 (\log_{1/2}(x^2 + 4x + 3) + \sin^{-1}[x]$ is defined if

$0 < x^2 + 4x + 3 < 1$ and $-1 \le x \le 2$ (I)

$x^2 + 4x + 3 > 0 \Rightarrow x > -1$ or $x < -3$ (II)

$x^2 + 4x + 3 < 1 \Rightarrow -2 - \sqrt{2} < x < -2 + \sqrt{2}$ (III)

So the domain of the function is $-1 < x < -2 + \sqrt{2}$.

70. $g(f(x)) = |\sin x| = \sqrt{\sin^2 x}$

which is satisfied if $f(x) = \sin^2 x$ and $g(x) = \sqrt{x}$

Also it satisfies $f(g(x)) = f\left(\sqrt{x}\right) = (\sin\sqrt{x})^2$

71. $f(x) = 3x - 5 = y \Rightarrow x = \dfrac{y+5}{3} = f^{-1}(y)$

$\Rightarrow f^{-1}(x) = \dfrac{x+5}{3}$

72. $f(x^2) = |x^2 - a| \neq (f(x))^2$

$f(|x|) = ||x| - a| \neq |f(x)|$

$f(x + y) = |x + y - a| \neq |x - a| + |y - a|$.

73. Required number is $4! = 24$.

Practise Questions - Part 1

1. R is not symmetric as $(2, 3) \in R$ but $(3, 2) \notin R$

2. We must have

$7 - x \geq 1$, $x - 3 \geq 0$ and $7 - x \geq x - 3$

$\Rightarrow$ $x \leq 6$, $x \geq 3$ and $x \leq 5$

Thus,

$3 \leq x \leq 5$

$\therefore$ Range of $f = \{{}^4P_0, {}^3P_1, {}^2P_2)$

$= \{1, 3, 2\}$

3. We have $-1 \leq x - 3 \leq 1$ and $9 - x^2 > 0$

$\Rightarrow$ $2 \leq x \leq 4$ and $-3 < x < 3$

$\therefore$ domain of f is $[2, 3,)$

4. We have

$-\sqrt{a^2+b^2} \leq a \sin x + b \cos x \leq \sqrt{a^2+b^2}$

Thus,

$-\sqrt{1+3} \leq \sin x - \sqrt{3} \cos x \leq \sqrt{1+3}$

$\therefore$ $-2 + 1 \leq f(x) \leq 2 + 1$ $\Rightarrow$ $-1 \leq f(x) \leq 3$

5. A function $f(x)$ is symmetrical about the line $x = a$ if

$$f(a-x) = f(a+x).$$

6. Relation R is clearly reflexive. Since $(6, 12) \in R$ but $(12, 6) \notin R$, R is not symmetric.

Also, as $(3,3) \in R$, $(3, 6) \in R \Rightarrow (3, 6) \in R$;

$(3, 3) \in R$, $(3, 9) \in R \Rightarrow (3, 9) \in R$;

$(3, 3) \in R$, $(3, 12) \in R \Rightarrow (3, 12) \in R$

$(6, 6) \in R$, $(6, 12) \in R \Rightarrow (6, 12) \in R$

Thus, R is transitive.

$\therefore$ R is reflexive and transitive.

7. Putting $x = y = 0$, we get

$$f(0) = f(0)\,f(0) - f(a)f(a)$$

$\Rightarrow$ $f(a)^2 = 1^2 - 1 = 0 \Rightarrow f(a) = 0$

Putting $x = a$, $y = a$, we get

$$f(0) = [f(a)]^2 - f(0)\,f(2a)$$

$$1 = 0 - f(2a) \Rightarrow f(2a) = -1$$

Next, we put $x = 0$, $y = a$ to obtain

$$f(-a) = f(0)\,f(a) - f(a)\,f(2a) = 0$$

Now, putting $x = 2a$, $y = x$, we get

$$f(2a - x) = f(2a)\,f(x) - f(-a)\,f(a + x)$$

$$= (-1)\,f(x) - (0)\,f(a + x) = -f(x)$$

8. Let $\omega \in W$, then $(\omega, \omega) \in R$.

$\therefore$ R is reflexive.

Also, if $\omega_1, \omega_2, \in W$ and $(\omega_1, \omega_2) \in R$, then $(\omega_2, \omega_1) \in$ R.

$\therefore$ R is symmetric.

Next, let ω_1 = ink ω_2 = link and ω_3, = let, then $(\omega_1, \omega_2) \in R$,

$(\omega_2, \omega_3) \in R$ but $(\omega_1, \omega_3) \notin R$.

Thus, R is not transitive.

9. 4^{-x^2} is defined for all $x \in R$.

$\cos^{-1}\left(\dfrac{x}{2} - 1\right)$ is defined if

$-1 \leq \dfrac{x}{2} - 1 \leq 1$ or for $0 \leq x \leq 4$

and log cos x is defined if

$\cos x > 0$ or for $2n\pi - \dfrac{\pi}{2} < x < 2n\pi + \dfrac{\pi}{2}$

where $n \in N$.

Thus, domain of f is $0 \leq x < \dfrac{\pi}{2}$ or $\left[0, \dfrac{\pi}{2}\right)$

10. As $x \neq x + 1$, S is not reflexive

$\Rightarrow$ S is not an equivalence relation.

As $x - x = 0 \in \mathbf{I}$, T is reflexive

$xTy \Rightarrow x - y \in \mathbf{I} \Rightarrow y - x \in \mathbf{I} \Rightarrow yTx$

$\therefore$ T is symmetric

Suppose xTy and yTz

$\Rightarrow x - y \in \mathbf{I}$, $y - z \in \mathbf{I}$

$\Rightarrow (x - y) + (y - z) \in \mathbf{I} \Rightarrow x - z \in \mathbf{I} \Rightarrow xTz$

$\therefore$ T is transitive

11. As f is one-one and onto, f is invertible. So $y = 4x + 3 \Rightarrow x = \dfrac{1}{4}(y - 3)$

Thus, inverse of f is $g(y) = \dfrac{1}{4}(y - 3)$.

12. $B = ((A \cup B) - A) \cup (A \cap B)$

$= ((A \cup C) - A) \cup (A \cap C) = C.$

13. Note that R is not symmetric as $(0, 1) \in$ R but $(1, 0) \notin$ R.

Next, note that $qm = pm \Leftrightarrow \frac{m}{n} = \frac{p}{q}$

Thus, $\left(\frac{m}{n}, \frac{p}{q}\right) \in S \Rightarrow \frac{m}{n} = \frac{p}{q}$

Clearly, S is reflexive and symmetric

Next, note that

$\left(\frac{m}{n}, \frac{p}{q}\right) \in S$ and $\left(\frac{p}{q}, \frac{r}{s}\right) \in S \Rightarrow \frac{m}{n} = \frac{p}{q}$ and $\frac{p}{q} = \frac{r}{s}$

$\Rightarrow \quad \frac{m}{n} = \frac{r}{s} \quad \Rightarrow \quad \left(\frac{m}{n}, \frac{r}{s}\right) \in \text{S}$

Thus S is transitive so is an equivalence relation.

14. A is *reflexive*

Let $x \in \mathbf{R}$. $(x, x) \in A$ as $x - x = 0$ is an integer.
A is *symmetric*

Suppose $(x, y) \in A \Rightarrow x - y$ is an integer

$\Rightarrow \quad y - x$ is an integer

$\Rightarrow \quad (y, x) \in A$

A is *transitive*

Let $(x, y) \in A$ and $(y, z) \in A$

$\Rightarrow x - y$ and $y - z$ are integers

$\Rightarrow (x - y) + (y - z)$ is an integer

$\Rightarrow x - z$ is an integer.

$\Rightarrow (x, z) \in A$

Thus, A is an equivalence relation.

B is not *symmetric*

Note that $(0, \sqrt{3}) \in B$ as $0 = (0)\sqrt{3}$ and 0 is a rational number, but $(\sqrt{3}, 0) \notin B$ as there is no rational number a such that $\sqrt{3} = (a)(0)$.

$\therefore$ Statement-1 is true but statement-2 is false. TIP As statement-2 is false, only possible answer is (*d*).

15. Let $y = f(x) = (x - 1)^2 + 1$

$\Rightarrow y - 1 = (x - 1)^2 \quad \Rightarrow x = 1 + \sqrt{y-1}, y \geq 1.$

Thus, $f^{-1}(x) = 1 + \sqrt{x-1}, x \geq 1.$

Now, $f(x) = f^{-1}(x)$

$\Rightarrow (x - 1)^2 + 1 = \sqrt{x-1} + 1$

$\Rightarrow \sqrt{x-1}\,[(x - 1)^{3/2} - 1] = 0$

$\Rightarrow x = 1, 2$

16. As $A = I^{-1} AI$, we get

$(A, A) \in R$ for all A.

$\therefore \quad R$ is reflexive.

Next, assume $(A, B) \in R$

$\Rightarrow$ there exists a non-singular matrix P such that $A = P^{-1}BP$

$\Rightarrow \quad B = PAP^{-1} = (P^{-1})^{-1}AP^{-1}$

Thus, $(B, A) \in R$

$\therefore \quad R$ is symmetric.

Next, assume that $(A, B) \in R$ and $(B, C) \in R$

$\Rightarrow$ there exist non-singular matrices P and Q such that

$A = P^{-1}BP$ and $B = Q^{-1}CQ$

$\therefore \quad A = P^{-1}(Q^{-1}CQ)P$

$= (P^{-1}Q^{-1})\,C(QP)$

$= (QP)^{-1}\,C(QP)$ [using statement-2]

$\Rightarrow (A, C) \in R$

$\therefore$ Statement-1 is true and statement-2 is true and is correct explanation for statement-1.

17. For each $x \in X$, we have the following four choices:

(a) $x \in Y$ and $x \in Z$ (b) $x \notin Y$ and $x \in Z$
(c) $x \in Y$ and $x \notin Z$ (d) $x \notin Y$ and $x \notin Z$

As we do not want $x \in Y \cap Z$, we are left with three choices for each $x \in X$.

Thus, the total number of ways in which Y and Z can be chosen is 3^5.

18. A $\times$ B contain (2) (4) = 8 elements.

The number of subsets of A $\times$ B having 3 or more elements
= Number of subsets of A $\times$ B – [number of subsets with at most 2 elements]

$= 2^8 - ({}^8C_1 + {}^8C_1 + {}^8C_2) = 219$

19. $P = \{(a, b) : \sec^2 a - \tan^2 b = 1\}$

P is not reflexive as $(\pi/2, \pi/2) \notin P$.

P is symmetric.

Suppose $(a, b) \in P$, then

$\sec^2 a - \tan^2 b = 1$

$\Rightarrow \quad (1 + \tan^2 a) - (\sec^2 b - 1) = 1$

$\Rightarrow \quad \sec^2 a - \tan^2 b = 1$

$\Rightarrow \quad (b, a) \in P.$

Suppose $(a, b) \in P, \quad (b, c) \in P$

$\Rightarrow \sec^2 a - \tan^2 b = 1$ and $\sec^2 b - \tan^2 c = 1$

Adding, we get

$\Rightarrow \quad \sec^2 a + \sec^2 b - \tan^2 b - \tan^2 c = 2$

$\Rightarrow \quad \sec^2 a + 1 - \tan^2 c = 2$

$\Rightarrow \quad \sec^2 a - \tan^2 c = 1$

$\therefore (a, c) \in P$.

20. $\frac{1}{3}+\frac{3n}{100}<1 \Rightarrow n \le 22;$

$1 \le \frac{1}{3}+\frac{3n}{100}<2 \quad \Rightarrow 300 \le 100 + 9n < 600$

$\Rightarrow \quad 23 \le n \le 55;$

Also, $\frac{1}{3}+\frac{3(56)}{100}=2\frac{1}{75}$

Thus, $\sum_{n=1}^{56} f(n) = \sum_{n=1}^{22} f(n) + \sum_{n=23}^{55} f(n) + f(56)$

$= 0 + \sum_{n=23}^{55} f(n) + 2(56)$

$= \frac{33}{2}(55+23)+112 = 1399$

21. Period $|\sin(4x)|$ is $\pi/4$ and period of $|\cos 2x|$ is $\pi/2$.

$\therefore$ period of $f(x) = |\sin 4x| + |\cos (2x)|$ is

$\left(l\text{cm}\left(\frac{1}{2},\frac{1}{4}\right)\right)\pi = \frac{\pi}{2}$

22. $f(-1) = f(1) = 0 \;\therefore f$ is not one-one.

Also $f(x) = \frac{|x|-1}{|x|+1} = 1 - \frac{2}{|x|+1} \le 1$ for all $x \in \mathbf{R}$

Thus f cannot be onto.

23. $A = \{-2, -1, 0, 1, 2\}$

$R = \{(-2, 2), (0, 0), (1, 1), (2, 2)\}$

As R has four elements, the power set of R contains 16 elements.

24. $n(A \times B) = n(A)n(B) = 4.2 = 8$

The number of subsets of $A \times B$ which contains at least three elements.

$= {}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6 + {}^8C_7 + {}^8C_8$

$= 2^8 - ({}^8C_0 + {}^8C_1 + {}^8C_2) = 256 - (1 + 8 + 28)$

$= 219$

25. Let total number of families in the town be x. Then $n(P) = 0.25x$, $n(C) = 0.15x$, $n(P' \cap C') = 0.65x$,

$n(P \cup C) = x - n(P' \cap C') = 0.35x$

i.e. 35% of the families own either a car or a phone.

$n(P \cup C) = n(P) + n(C) - n(P \cap C)$

$\Rightarrow 0.35x = 0.25x + 0.15x - n(P \cap C)$

$\Rightarrow n(P \cap C) = 0.05x$

i.e. 5% of the families own a car and a phone

As $0.05x = 2000 \Rightarrow x = 40{,}000$

Thus, (a), (b), (c) are all true.

26. We can choose three elements out of seven in ${}^7C_3$ ways. These three elements are mapped to y_2. The remaining 4 elements are to be mapped to y_1 and y_3. This can be done is 2^4 ways. But these include two ways in which all the four elements are mapped to either y_1 or y_3. Therefore, there are $2^4 - 2 = 14$ ways to map containing four elements to y_1 and y_3. Thus the required number of onto mappings is $14.{}^7C_3$.

27. $f(x) + 2f\left(\frac{1}{x}\right) = 3x \;\forall\; x \ne 0$ (i)

Replacing x by $\frac{1}{x}$, we have

$f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x}$ (ii)

(i) – 2(ii) gives $f(x) - 4f(x) = 3x - \frac{6}{x}$

$\Rightarrow f(x) = \frac{2}{x} - x$

$f(x) = f(-x) \Rightarrow \frac{2}{x} - x = -\frac{2}{x} + x \Rightarrow \frac{4}{x} = 2x \Rightarrow x^2 = 2$

$x = \pm\sqrt{2}$. Thus S has exactly two elements.

28. Let $y = 3^{x(x-1)}$, so $y \ge 1$

$\Rightarrow \log_3 y = x(x - 1)$

$\Rightarrow x^2 - x - \log_3 y = 0$

$\Rightarrow x = \frac{1 \pm \sqrt{1+4\log_3 y}}{2}$. As $x \ge 1$, so

$x = \frac{1}{2}\left[1+\sqrt{1+4\log_3 y}\right]$.

Thus $f^{-1}(x) = \frac{1}{2}\left[1+\sqrt{1+4\log_3 y}\right]$.

29. $f_1(x) = f_0(f_0(x)) = \frac{1}{1-f_0(x)} = \frac{1}{1-\frac{1}{1-x}}$

$= \frac{1-x}{1-x-1} = 1-\frac{1}{x}$

$f_2(x) = f_0(f_1(x)) = \frac{1}{1-f_1(x)} = \frac{1}{1-\left(1-\frac{1}{x}\right)} = x$

$f_3(x) = f_0(f_2(x)) = f_0(x)$

Thus $f_{3k}(x) = f_0(x)$, $f_{3k+1}(x) = f_1(x)$,

$f_{3k+2}(x) = f_2(x)$ for all $x \in \mathbf{R} \sim \{0, 1\}, k \geq 0$

So, $f_{100}(3) + f_1\left(\frac{2}{3}\right) + f_2\left(\frac{3}{2}\right)$

$= f_1(3) + f_1\left(\frac{2}{3}\right) + f_2\left(\frac{3}{2}\right)$

$= 1-\frac{1}{3}+1-\frac{3}{2}+\frac{3}{2}= \frac{5}{3}.$

Practise Questions - Part 2

1. f is defined if $2x - 3 \geq 0, x \geq 1$

   So the domain of f is $\left[\frac{3}{2}, \infty\right)$.

2. If $\frac{x_1+2}{x_1-1} = \frac{x_2+2}{x_2-1} \Rightarrow x_1x_2 + 2x_2 - x_1 - 2$

   $= x_1x_2 + 2x_1 - x_2 - 2$

   $\Rightarrow 3x_2 = 3x_1 \Rightarrow x_1 = x_2$

   So f is one-one

   If $y = f(x) = \frac{x+2}{x-1} \Rightarrow yx - y = x + 2$

   $\Rightarrow (y - 1)x = y + 2$

   $\Rightarrow x = \frac{y+2}{y-1} \in (1, \infty)$

   Thus f is onto

3. $x^6 + y^6 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$, so if $(x, y) \in A \cap B$ then $1 = x^4 - x^2y^2 + y^4 = (x^2 + y^2)^2 - 3x^2y^2$

   $= 1 - 3x^2y^2$

   $\Rightarrow x^2y^2 = 0 \Rightarrow x = 0$ or $y = 0$ but $x > 0, y > 0$

   so $A \cap B = \phi$

4. $58 = n(F \cup B \cup C)$

   $= n(f) + n(B) + n(C) - n(F \cap B) - n(B \cap C) - n(F \cap C) + n(F \cap B \cap C)$

   $= 38 + 15 + 20 - n(F \cap B) - n(B \cap C) - n(F \cap C) + 3$

   $\Rightarrow n(F \cap B) + n(B \cap C) + n(F \cap C) = 18$

   Number of medals received in exactly two of three sports is

   $= n(F \cap B \cap C') + n(B \cap C \cap F') + n(F \cap C \cap B')$

   $= n(F \cap B) + n(B \cap C) + n(F \cap C) - 3\, n(F \cap \mathrm{B} \cap C)$

   $= 18 - 9 = 9$

5. Since $1 + x^2 > 0$ for all $x \in Q$, so R is reflexive. If $(x, y) \in R \Rightarrow 1 + xy > 0$

   $\Rightarrow 1 + yx > 0 \Rightarrow (y, x) \in R$

   Thus R is symmetric. Take $x = 2, y = -\frac{1}{4}, z = -1$

   $1 + xy = 1 - \frac{1}{2} = \frac{1}{2} > 0$ so $(x, y) \in R$

   $1 + yz = 1 + \frac{1}{4} = \frac{5}{4} > 0$ so $(y, z) \in R$

   $1 + xz = 1 - 2 = -1 \Rightarrow (x, z) \notin R$. Hence R is not transitive.

6. f is defined if $x > 3$ and $3 \neq \log_3(x - 3)$

   $\Rightarrow x > 3, \ x - 3 \neq 27$ i.e. $x \in (3, \infty) \sim \{30\}$

   $= (3, 30) \cup (30, \infty\}$

7. f is continuous function and $f'(x) = 2009\, x^{2008} + 2009 > 0$. So f is one-one and onto.

8. For $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $0 \leq \cos x \leq 1$. So

   $0 \leq \cos^4 x \leq 1$ and $-5 \leq -2\cos^2 x - 2\cos x - 1 \leq -1$

   $\Rightarrow -4 \leq \cos^4 x - 2\cos^2 x - 2\cos x - 1 \leq -1$

   $\Rightarrow -12 \leq f(x) \leq -3$

9. $f(-x) = (-x)^2\, e^{-(-x)^2} \sin|-x| = x^2 e^{-x^2} \sin|x| = f(x)$. Product of two odd functions is an even function.

10. $x = 1\cdot x$ so $(x, x) \in R_1$. If $(x, y) \in R_1$ then $x = ay$ or $y = ax$ for some integer a

    $\Rightarrow y = ax$ or $x = ay \Rightarrow (y, x) \in R_1$

    If $x = ay$ or $y = ax$ and $z = by$ or $y = bz$

    for some integer a and b then $x = abz$ or $z = abx$, so $(x, z) \in R_1$. Thus R_1 is an equivalence relation.

    Since $(2, 2) \notin R_2$ so R_2 is not reflexive. Thus R_2 is not an equivalence relation.

11. $\text{gof}(x) = g\left(\frac{1}{x+1}\right) = \left(\frac{1}{x+1}\right)^2 + 1 = \frac{1+(x+1)^2}{(x+1)^2} \geq 1$

    so gof is not onto, gof is not one-one as gof (1) = gof (–3).

12. Since g: $cd\,(b, c) = 1$ so $bc = \text{l.c.m}\,(b, c)$. Hence $d = bc$

13. $y = f(x) = (x + 1)^2 - 1 \Rightarrow x = \sqrt{y+1} - 1$

    Hence $f^{-1}(x) = \sqrt{x+1} - 1$. Now $f(x) = f^{-1}(x)$

    $\Leftrightarrow (x + 1)^2 - 1 = \sqrt{x+1} - 1 \Rightarrow (x + 1)^4 = x + 1$

    $\Rightarrow (x + 1)\,[x^3 + 3x^2 + 3x] = 0$

$\Rightarrow (x + 1)\, x(x^2 + 3x + 3) = 0$

$\Rightarrow x = 0, -1$

14. For $x < 0$, $f(x) = \dfrac{e^{-x} - e^{-x}}{e^{x} + e^{-x}} = 0$

$\therefore$ f cannot be one-one

For $x \geq 0$, $f(x) = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \geq 0$

Thus $f(x)$ cannot take negative values so range of f cannot be **R**. Therefore f is not onto.

15. $f(x + 4) - f(x + 2) + f(x) = 0 \ \forall\ x \in \mathbf{R}$

Replace x by $x + 2$,

$f(x + 6) - f(x + 4) + f(x + 2) = 0$

Adding above two equations, we have

$f(x + 6) + f(x) = 0 \ \forall\ x \in \mathbf{R}$

Replace x by $x + 6$, we get

$f(x + 12) + f(x + 6) = 0$

$\Rightarrow f(x + 12) = -f(x + 6) = f(x)$

f is periodic with period 12.